

Arithmetic Sequences

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A succession of numbers which are arranged in a specific order according to a rule is called a *sequence*. Each number in a sequence is called a *term* and the number from the beginning is called the n^{th} *term*. (For example, the initial term is called the 1st *term*.)

1. Fill in the blanks with the appropriate number for each given sequence.

(1) 2, 5, 8, 11, 14, 17, ...

(2) 2, 4, 8, 16, 32, 64, ...

(3) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, ...

(4) 4, 2, 0, -2, -4, -6, ...

(5) 1, 4, 9, 16, 25, 36, ...

2. Find the terms for each given sequence where the n^{th} term is expressed by substituting $n=1, 2, 3, \dots, 6$.

	1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term
$3n-1$	2	5	8	11	14	17
(1) 2^n	2	4	8	16	32	64
(2) $\frac{n}{n+1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
(3) $6-2n$	4	2	0	-2	-4	-6
(4) n^2	1	4	9	16	25	36
(5) $(-1)^n$	-1	1	-1	1	-1	1

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numbers which are arranged in a specific order according to
sequence. Each number in a sequence is called a *term* and the
 beginning is called the *nth term*. (For example, the initial term
)

s with the appropriate number for each given sequence.

1, 14, 17, ...

6, 32, 64, ...

$\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, ...

-2, -4, -6, ...

, 25, 36, ...

for each given sequence where the *nth* term is expressed
 substituting *n* = 1, 2, 3, ..., 6.

1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term
2	5	8	11	14	17
2	4	8	16	32	64
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
4	2	0	-2	-4	-6
1	4	9	16	25	36
-1	1	-1	1	-1	1

Generally, a sequence is expressed as follows:

$$\begin{array}{ccccccc} a_1, & a_2, & a_3, & \dots, & a_n, & \dots \\ \uparrow & \uparrow & \uparrow & & \uparrow & \\ 1^{\text{st}} \text{ term} & 2^{\text{nd}} \text{ term} & 3^{\text{rd}} \text{ term} & \dots & n^{\text{th}} \text{ term} & \dots \end{array}$$

This sequence is also expressed as $\{a_n\}$.

When the n^{th} term a_n of a sequence $\{a_n\}$ is expressed in terms of n , it is called the **general term** of $\{a_n\}$.

3. Choose the general term of each given sequence $\{a_n\}$ from (A)~(H) and find the 10th term.

Ex. 3, 5, 7, 9, 11, ...

[Sol] $a_n: (A) \quad a_{10} = 21$

(1) 7, 4, 1, -2, -5, ...

[Sol] $a_n: (E) \quad a_{10} = -20$

(2) 2, 4, 8, 16, 32, ...

[Sol] $a_n: (C) \quad a_{10} = 1024$

(3) 1, -1, 1, -1, 1, ...

[Sol] $a_n: (F) \quad a_{10} = -1$

(4) $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \dots$

[Sol] $a_n: (H) \quad a_{10} = \frac{10}{13}$

(A) $2n+1$ (B) $-2n+9$ (C) 2^n (D) $(-1)^n$

(E) $-3n+10$ (F) $-(-1)^n$ (G) $\frac{n}{2n+2}$ (H) $\frac{n}{n+3}$

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A sequence whose terms are found by successively adding a fixed number to the 1st term a is called an *arithmetic sequence* (or *arithmetic progression*). The fixed number d is called the *common difference* of the arithmetic sequence.

1. Find the 10th term of the following arithmetic sequence $\{a_n\}$.

10, 13, 16, 19, 22, ...

[Sol] $a_1 = 10$

$$a_2 = 10 + 1 \cdot 3 = 13$$

$$a_3 = 10 + 2 \cdot 3 = 16$$

$$a_4 = 10 + 3 \cdot 3 = 19$$

⋮

$$a_{10} = 10 + 9 \cdot 3 = 37$$

2. Find the general term of the following arithmetic sequence $\{a_n\}$.

8, 4, 0, -4, -8, ...

[Sol] $a_1 = 8$

$$a_2 = 8 + 1 \cdot (-4) = 4$$

$$a_3 = 8 + 2 \cdot (-4) = 0$$

$$a_4 = 8 + 3 \cdot (-4) = -4$$

⋮

$$a_{n-2} = 8 + (n-2) \cdot (-4) = -4n + 16$$

$$a_n = 8 + (n-1) \cdot (-4) = -4n + 12$$

3. Find the general term of the following arithmetic sequence $\{a_n\}$ with first term a and common difference d .

[Sol] $a_1 = a$

$$a_2 = a + 1d$$

$$a_3 = a + 2d$$

$$a_4 = a + 3d$$

⋮

$$a_n = a + (n-1)d \quad \leftarrow$$

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ +d & +d & +d & \dots & +d & +d \end{array}$$

The number of $+d$ is $(n-1)$.

General Term of an Arithmetic Sequence

The general term of an arithmetic sequence $\{a_n\}$ with 1st term a and common difference d is

$$a_n = a + (n-1)d$$

Ex. Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -2 and 6th term is 8.

[Sol] $a_6 = a + 5 \cdot (-2) = 8$

$$\therefore a = 18$$

$$\text{Also, } a_n = 18 + (n-1) \cdot (-2) = -2n + 20$$

Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -3 and 11th term is -27 .

[Sol] $a_{11} = a + 10 \cdot (-3) = -27$

$$\therefore a = 3$$

$$\text{Also, } a_n = 3 + (n-1) \cdot (-3) = -3n + 6$$

Find the common difference d and the general term of the arithmetic sequence $\{a_n\}$ whose 1st term is 20 and 15th term is 90.

[Sol] $a_{15} = 20 + 14d = 90$

$$\therefore d = 5$$

$$\text{Also, } a_n = 20 + (n-1) \cdot 5 = 5n + 15$$

Given the arithmetic sequence $\{a_n\}$ whose 1st term is -25 and common difference is 5, find the value of n for which the n th term is -5 .

[Sol] $a_n = -25 + (n-1) \cdot 5 = 5n - 30$

Since $a_n = -5$,

$$5n - 30 = -5$$

$$\therefore n = 5$$

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Ex.

Find the general term of the arithmetic sequence $\{a_n\}$ whose 5th term is -5 and 9th term is 11 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_5 = a + 4d = -5 & \cdots \textcircled{1} \\ a_9 = a + 8d = 11 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -21, d = 4$$

$$\therefore a_n = -21 + (n-1) \cdot 4 = 4n - 25$$

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose 5th term is -3 and 14th term is 24 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_5 = a + 4d = -3 & \cdots \textcircled{1} \\ a_{14} = a + 13d = 24 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -15, d = 3$$

$$\therefore a_n = -15 + (n-1) \cdot 3 = 3n - 18$$

2. Find the general term of the arithmetic sequence $\{a_n\}$ whose 3rd term is 7 and 8th term is -8 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_3 = a + 2d = 7 & \cdots \textcircled{1} \\ a_8 = a + 7d = -8 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 13, d = -3$$

$$\therefore a_n = 13 + (n-1) \cdot (-3) = -3n + 16$$

3. Find the 55th term of the arithmetic sequence $\{a_n\}$ whose 15th term is 77 and 42nd term is 239.

Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_{15} = a + 14d = 77 & \dots \textcircled{1} \\ a_{42} = a + 41d = 239 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -7, d = 6$$

$$\therefore a_{55} = -7 + 54 \cdot 6 = 317$$

4. Given the arithmetic sequence $\{a_n\}$ whose 53rd term is -47 and 77th term is -95 , find the value of n for which the n^{th} term is -111 .

Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_{53} = a + 52d = -47 & \dots \textcircled{1} \\ a_{77} = a + 76d = -95 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 57, d = -2$$

$$\therefore a_n = 57 + (n-1) \cdot (-2) = -2n + 59$$

Since $a_n = -111$,

$$-2n + 59 = -111$$

$$\therefore n = 85$$

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Ex. Find the value of x when the sequence 1, x , 8, ... is an arithmetic sequence.

[Sol] Since the sequence 1, x , 8 is an arithmetic sequence,

$$x - 1 = 8 - x$$

$$\therefore x = \frac{9}{2}$$

1. Find the value of x when the sequence $\frac{5}{6}$, x , $\frac{5}{2}$, ... is an arithmetic sequence.

[Sol] Since the sequence $\frac{5}{6}$, x , $\frac{5}{2}$ is an arithmetic sequence,

$$x - \frac{5}{6} = \frac{5}{2} - x$$

$$\therefore x = \frac{5}{3}$$

2. Find the value of x when the sequence x , 6, $2x$, ... is an arithmetic sequence.

[Sol] Since the sequence x , 6, $2x$ is an arithmetic sequence,

$$6 - x = 2x - 6$$

$$\therefore x = 4$$

3. Find the values of x and y when the sequence 1, x , y , -11 , ... is an arithmetic sequence.

[Sol] Since the sequence 1, x , y , -11 is an arithmetic sequence,

$$\begin{cases} x - 1 = y - x & \cdots \textcircled{1} \\ y - x = -11 - y & \cdots \textcircled{2} \end{cases}$$

From ① and ②,

$$x = -3, y = -7$$

N4b

Ex Prove that the sequence $\{a_n\}$ where $a_n = 5n + 1$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n = 5n + 1$, $a_{n+1} = 5(n+1) + 1 = 5n + 6$

$$\therefore a_{n+1} - a_n = (5n + 6) - (5n + 1) = 5$$

Since $a_{n+1} - a_n$ is fixed as 5,
 $\{a_n\}$ is an arithmetic sequence.

Also, $a_1 = 5 \cdot 1 + 1 = 6$

Therefore, the 1st term is 6 and the common difference is 5.

If $a_{n+1} - a_n = d$ (fixed),
then it is an arithmetic
sequence.

4. Prove that the sequence $\{a_n\}$ where $a_n = 4n + 3$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n = 4n + 3$, $a_{n+1} = 4(n+1) + 3 = 4n + 7$

$$\therefore a_{n+1} - a_n = (4n + 7) - (4n + 3) = 4$$

Since $a_{n+1} - a_n$ is fixed as 4, $\{a_n\}$ is an arithmetic sequence.

Also, $a_1 = 4 \cdot 1 + 3 = 7$

Therefore, the 1st term is 7 and the common difference is 4.

5. Given that p and q are constants, prove that the sequence $\{a_n\}$ where $a_n = pn + q$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n = pn + q$, $a_{n+1} = p(n+1) + q = pn + p + q$

$$\therefore a_{n+1} - a_n = (pn + p + q) - (pn + q) = p$$

Since $a_{n+1} - a_n$ is fixed as p , $\{a_n\}$ is an arithmetic sequence.

Also, $a_1 = p \cdot 1 + q = p + q$

Therefore, the 1st term is $p + q$ and the common difference is p .

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Ex.

Find the three numbers whose sum is 15 and product is 80, which form an arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d$, c , $c+d$,

$$\begin{cases} (c-d) + c + (c+d) = 15 & \dots \textcircled{1} \\ (c-d)c(c+d) = 80 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$c = 5, d = \pm 3$$

Therefore, the three numbers are: 2, 5, 8

From $\textcircled{1}$, $c = 5$
Substituting this in

2, 5, 8 when
and $d = -3$

1. Find the three numbers whose sum is 27 and product is 693, which form an arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d$, c , $c+d$,

$$\begin{cases} (c-d) + c + (c+d) = 27 & \dots \textcircled{1} \\ (c-d)c(c+d) = 693 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$c = 9, d = \pm 2$$

Therefore, the three numbers are: 7, 9, 11

N5b

2. Find the 1st term and the common difference of the arithmetic sequence $\{a_n\}$, given $a_1 + a_2 + a_3 = -15$ and $a_1 a_2 a_3 = 120$.

[Sol] Let c be a_2 and d be the common difference.

Since the three numbers are expressed as $a_1 = c - d$, $a_2 = c$, $a_3 = c + d$,

$$\begin{cases} (c-d) + c + (c+d) = -15 & \dots ① \\ (c-d)c(c+d) = 120 & \dots ② \end{cases}$$

From ① and ②,

$$c = -5, d = \pm 7$$

$$\text{When } d = 7, a_1 = -5 - 7 = -12$$

$$\text{When } d = -7, a_1 = -5 - (-7) = 2$$



The 1st term varies with d ; therefore, consider the different cases of d .

Therefore, the 1st term is -12 and the common difference is 7 ; or

the 1st term is 2 and the common difference is -7 .

3. Find the 1st term and the common difference of the arithmetic sequence $\{a_n\}$, given $a_1 + a_3 + a_5 = -12$ and $a_1 a_3 a_5 = 80$.

[Sol] Let c be a_3 and d be the common difference.

Since the three numbers are expressed as $a_1 = c - 2d$, $a_3 = c$, $a_5 = c + 2d$,

$$\begin{cases} (c-2d) + c + (c+2d) = -12 & \dots ① \\ (c-2d)c(c+2d) = 80 & \dots ② \end{cases}$$

From ① and ②,

$$c = -4, d = \pm 3$$

$$\text{When } d = 3, a_1 = -4 - 2 \cdot 3 = -10$$

$$\text{When } d = -3, a_1 = -4 - 2 \cdot (-3) = 2$$

Therefore, the 1st term is -10 and the common difference is 3 ; or

the 1st term is 2 and the common difference is -3 .

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Ex.

Find the sum S_6 of the arithmetic sequence whose 1st term is 1, common difference is 2 and number of terms is 6.

$$\begin{aligned}
 \text{[Sol]} \quad S_6 &= 1 + 3 + 5 + 7 + 9 + 11 \\
 +) \quad S_6 &= 11 + 9 + 7 + 5 + 3 + 1 \\
 \hline
 2S_6 &= 12 + 12 + 12 + 12 + 12 + 12 \\
 \therefore 2S_6 &= 12 \cdot 6 \\
 \therefore S_6 &= 36
 \end{aligned}$$

$$S_6 = a_1 + a_2 + a_3 + \cdots + a_6$$

$$S_6 = a_6 + a_5 + a_4 + \cdots + a_1$$

S_6 will be the same even if the order of terms is reversed.

1. Find the sum S_8 of the arithmetic sequence whose 1st term is 1, common difference is 4 and number of terms is 8.

$$\begin{aligned}
 \text{[Sol]} \quad S_8 &= 1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 \\
 +) \quad S_8 &= 29 + 25 + 21 + 17 + 13 + 9 + 5 + 1 \\
 \hline
 2S_8 &= 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 \\
 \therefore 2S_8 &= 30 \cdot 8 \\
 \therefore S_8 &= 120
 \end{aligned}$$

Find the sum S_n of the arithmetic sequence with 1st term a , common difference d , last term l and number of terms n .

$$\begin{aligned}
 \text{[Sol]} \quad S_n &= a + (a+d) + (a+2d) + \cdots + (l-d) + l \\
 +) \quad S_n &= l + (l-d) + (l-2d) + \cdots + (a+d) + a \\
 \hline
 2S_n &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)
 \end{aligned}$$

The number of $(a+l)$ is n .

$$\therefore 2S_n = n(a+l)$$

$$\therefore S_n = \frac{1}{2} n(a+l)$$

Also, since l is the n^{th} term, $l = a + (n-1)d$

$$\therefore S_n = \frac{1}{2} n[2a + (n-1)d]$$

Sum of an Arithmetic Sequence

Let S_n be the sum of an arithmetic sequence with 1st term a , common difference d , last term l and number of terms n .

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a + (n-1)d]$$

2. Using the formula above, find the sum of each given arithmetic sequence.

Ex. 1st term: 3, last term: 21,
number of terms: 10

[Sol] $\frac{1}{2} \cdot 10(3+21) = 120$

1st term: 5, common difference: 4,
number of terms: 12

[Sol] $\frac{1}{2} \cdot 12[2 \cdot 5 + (12-1) \cdot 4] = 324$

(1) 1st term: 2, last term: 34,
number of terms: 13

[Sol] $\frac{1}{2} \cdot 13(2+34) = 234$

(3) 1st term: 4, common difference: 3,
number of terms: 15

[Sol] $\frac{1}{2} \cdot 15[2 \cdot 4 + (15-1) \cdot 3] = 375$

(2) 1st term: -3, last term: -42,
number of terms: 20

[Sol] $\frac{1}{2} \cdot 20[-3 + (-42)] = -450$

(4) 1st term: -2, common difference: 5,
number of terms: 18

[Sol] $\frac{1}{2} \cdot 18[2 \cdot (-2) + (18-1) \cdot 5] = 729$

Ex. S_7 , the sum of the first 7 terms of the arithmetic sequence 5, 9, 13, 17, ...

[Sol] $S_7 = \frac{1}{2} \cdot 7[2 \cdot 5 + (7-1) \cdot 4] = 119$

1st term: 5, common difference: 4,
number of terms: 7

(5) S_{11} , the sum of the first 11 terms of the arithmetic sequence -9, -3, 3, 9, ...

[Sol] $S_{11} = \frac{1}{2} \cdot 11[2 \cdot (-9) + (11-1) \cdot 6] = 231$

(6) S_{20} , the sum of the first 20 terms of the arithmetic sequence 50, 47, 44, 41, ...

[Sol] $S_{20} = \frac{1}{2} \cdot 20[2 \cdot 50 + (20-1) \cdot (-3)] = 430$

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Ex.

Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of first 4 terms is 80 and whose sum of the first 8 terms is 320.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_4 = \frac{1}{2} \cdot 4[2a + (4-1)d] = 80 & \dots \textcircled{1} \\ S_8 = \frac{1}{2} \cdot 8[2a + (8-1)d] = 320 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 5, d = 10$$

$$\therefore a_n = 5 + (n-1) \cdot 10 = 10n - 5$$

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 20 and whose sum of the first 7 terms is 7.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2} \cdot 5[2a + (5-1)d] = 20 & \dots \textcircled{1} \\ S_7 = \frac{1}{2} \cdot 7[2a + (7-1)d] = 7 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 10, d = -3$$

$$\therefore a_n = 10 + (n-1) \cdot (-3) = -3n + 13$$

N7b

2. Find the sum S of the first 30 terms of the arithmetic sequence $\{a_n\}$ whose sum of the first 10 terms is 100 and whose sum of the first 20 terms is 350.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_{10} = \frac{1}{2} \cdot 10[2a + (10-1)d] = 100 \quad \dots \textcircled{1} \\ S_{20} = \frac{1}{2} \cdot 20[2a + (20-1)d] = 350 \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = \frac{13}{4}, d = \frac{3}{2}$$

$$\therefore S = \frac{1}{2} \cdot 30 \left[2 \cdot \frac{13}{4} + (30-1) \cdot \frac{3}{2} \right] = 750$$

3. Find the sum S from the 11th to the 20th term of the arithmetic sequence $\{a_n\}$ whose 8th term is 37 and 24th term is 117.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} a_8 = a + 7d = 37 \quad \dots \textcircled{1} \\ a_{24} = a + 23d = 117 \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 2, d = 5$$

$$\therefore S_{20} = \frac{1}{2} \cdot 20[2 \cdot 2 + (20-1) \cdot 5] = 990$$

$$S_{10} = \frac{1}{2} \cdot 10[2 \cdot 2 + (10-1) \cdot 5] = 245$$

$$\therefore S = S_{20} - S_{10} = 990 - 245 = 745$$

$$\begin{array}{c} \overbrace{a_1 + a_2 + \dots + a_{10}}^{245} + \overbrace{a_{11} + \dots + a_{20}}^S \\ \hline 990 \end{array}$$

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Ex.

Given the arithmetic sequence $\{a_n\}$ whose 1st term is 53 and common difference is -6 , find the value of m for which the sum of the first terms is the greatest. Then, find the sum S .

[Sol] $a_n = 53 + (n-1) \cdot (-6) = -6n + 59$

$$-6n + 59 > 0 \text{ when}$$

$$n < \frac{59}{6} = 9.8\dots$$

m is the greatest natural number that satisfies this.

$$\therefore m = 9$$

$$\text{Also, } S = \frac{1}{2} \cdot 9 [2 \cdot 53 + (9-1) \cdot (-6)] = 261$$

In this arithmetic sequence

$$\underbrace{53, 47, \dots, 5}_{a_n > 0}, \underbrace{-1, \dots}_{a_n < 0}$$

The sum is the greatest when $53 + 47 + \dots + 5$.

1. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 58 and common difference is -3 , find the value of m for which the sum of the first m terms is the greatest. Then, find the sum S .

[Sol] $a_n = 58 + (n-1) \cdot (-3) = -3n + 61$

$$-3n + 61 > 0 \text{ when}$$

$$n < \frac{61}{3} = 20.3\dots$$

m is the greatest natural number that satisfies this.

$$\therefore m = 20$$

$$\text{Also, } S = \frac{1}{2} \cdot 20 [2 \cdot 58 + (20-1) \cdot (-3)] = 590$$

2. Given the arithmetic sequence $\{a_n\}$ whose 1st term is -120 and common difference is 7 , find the value of m for which the sum of the first m terms is the smallest. Then, find the sum S .

[Sol] $a_n = -120 + (n-1) \cdot 7 = 7n - 127$

$7n - 127 < 0$ when

$n < \frac{127}{7} = 18.1\dots$

m is the greatest natural number that satisfies this.

$\therefore m = 18$

Also, $S = \frac{1}{2} \cdot 18[2 \cdot (-120) + (18-1) \cdot 7] = -1089$

In this arithmetic sequence,

$\underbrace{-120, -113, \dots, -1}_{a_n < 0}, \underbrace{6, 13, \dots}_{a_n > 0}$

The sum is the smallest when $(-120) + (-113) + \dots + (-1)$.

3. Given the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is -445 and whose sum of the first 10 terms is -765 , find the value of m for which the sum of the first m terms is the smallest. Then, find the sum S .

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$\begin{cases} S_5 = \frac{1}{2} \cdot 5[2a + (5-1)d] = -445 & \dots \textcircled{1} \\ S_{10} = \frac{1}{2} \cdot 10[2a + (10-1)d] = -765 & \dots \textcircled{2} \end{cases}$

From $\textcircled{1}$ and $\textcircled{2}$,

$a = -99, d = 5$

$\therefore a_n = -99 + (n-1) \cdot 5 = 5n - 104$

$5n - 104 < 0$ when

$n < \frac{104}{5} = 20.8$

m is the greatest natural number that satisfies this.

$\therefore m = 20$

Also, $S = \frac{1}{2} \cdot 20[2 \cdot (-99) + (20-1) \cdot 5] = -1030$

Arithmetic Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex.

Find the sum S of all natural numbers from 10 to 100 which are divisible by 3.

[Sol] The general term of the sequence of natural numbers greater than or equal to 10 which are divisible by 3 is:

$$12 + (n-1) \cdot 3 = 3n + 9 \quad \leftarrow \text{1st term: 12, common difference: 3}$$

$$3n + 9 \leq 100 \text{ when } n \leq \frac{91}{3} = 30.3\dots$$

The greatest natural number n that satisfies this is $n = 30$.

$$\therefore S = \frac{1}{2} \cdot 30 [2 \cdot 12 + (30-1) \cdot 3] = 1665$$

1. Find the sum S of all natural numbers from 30 to 100 which are divisible by 6.

[Sol] The general term of the sequence of natural numbers greater than or equal to 30 which are divisible by 6 is:

$$30 + (n-1) \cdot 6 = 6n + 24$$

$$6n + 24 \leq 100 \text{ when } n \leq \frac{76}{6} = 12.6\dots$$

The greatest natural number n that satisfies this is $n = 12$.

$$\therefore S = \frac{1}{2} \cdot 12 [2 \cdot 30 + (12-1) \cdot 6] = 756$$

2. Find the sum S of all natural numbers from 10 to 200 which leave a remainder of 4 when divided by 7.

[Sol] The general term of the sequence of natural numbers greater than or equal to 10 which leave a remainder of 4 when divided by 7 is:

$$11 + (n-1) \cdot 7 = 7n + 4$$

$$7n + 4 \leq 200 \text{ when } n \leq \frac{196}{7} = 28$$

The greatest natural number n that satisfies this is $n = 28$.

$$\therefore S = \frac{1}{2} \cdot 28 [2 \cdot 11 + (28-1) \cdot 7] = 2954$$

N9b

3. Find the sum S of all natural numbers from 100 to 200 which are divisible by either 3 or 5.

[Sol] The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 3 is:

$$102 + (n-1) \cdot 3 = 3n + 99$$

$$3n + 99 \leq 200 \text{ when } n \leq \frac{101}{3} = 33.6\dots$$

The greatest natural number n that satisfies this is $n = 33$.

Therefore, the sum of natural numbers from 100 to 200 which are divisible by 3 is:

$$\frac{1}{2} \cdot 33 [2 \cdot 102 + (33-1) \cdot 3] = 4950 \dots \textcircled{1}$$

The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 5 is:

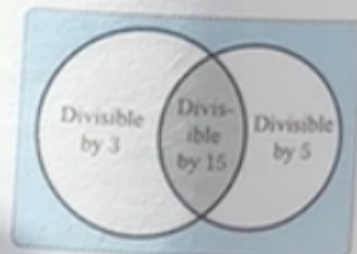
$$100 + (n-1) \cdot 5 = 5n + 95$$

$$5n + 95 \leq 200 \text{ when } n \leq \frac{105}{5} = 21$$

The greatest natural number n that satisfies this is $n = 21$.

Therefore, the sum of natural numbers from 100 to 200 which are divisible by 5 is:

$$\frac{1}{2} \cdot 21 [2 \cdot 100 + (21-1) \cdot 5] = 3150 \dots \textcircled{2}$$



The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 15 is:

$$105 + (n-1) \cdot 15 = 15n + 90$$

$$15n + 90 \leq 200 \text{ when } n \leq \frac{110}{15} = 7.3\dots$$

The greatest natural number n that satisfies this is $n = 7$.

Therefore, the sum of natural numbers from 100 to 200 which are divisible by 15 is:

$$\frac{1}{2} \cdot 7 [2 \cdot 105 + (7-1) \cdot 15] = 1050 \dots \textcircled{3}$$

From ①~③,

$$S = 4950 + 3150 - 1050 = 7050$$

Since ③ is included in both ① and ②, ③ has to be subtracted when finding S .

Arithmetic Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose 3rd term and 8th term is 50.

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_3 = a + 2d = 10 \cdots \textcircled{1} \\ a_8 = a + 7d = 50 \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -6, d = 8$$

$$\therefore a_n = -6 + (n-1) \cdot 8 = 8n - 14$$

2. Find the three numbers whose sum is 18 and product is 162, which form an arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d$, c , $c+d$,

$$\begin{cases} (c-d) + c + (c+d) = 18 \cdots \textcircled{1} \\ (c-d)c(c+d) = 162 \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$c = 6, d = \pm 3$$

Therefore, the three numbers are: 3, 6, 9

3. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 100 and whose sum of the first 10 terms is 150. $\Rightarrow$ N7

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2} \cdot 5[2a + (5-1)d] = 100 & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} S_{10} = \frac{1}{2} \cdot 10[2a + (10-1)d] = 150 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 24, d = -2$$

$$\therefore a_n = 24 + (n-1) \cdot (-2) = -2n + 26$$

4. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 50 and common difference is -3 , find the value of m for which the sum of the first m terms is the greatest. Then, find the sum S . $\Rightarrow$ N8

$$[\text{Sol}] a_n = 50 + (n-1) \cdot (-3) = -3n + 53$$

$$-3n + 53 > 0 \text{ when}$$

$$n < \frac{53}{3} = 17.6\dots$$

m is the greatest natural number that satisfies this.

$$\therefore m = 17$$

$$\text{Also, } S = \frac{1}{2} \cdot 17[2 \cdot 50 + (17-1) \cdot (-3)] = 442$$

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

A sequence whose terms are found by successively multiplying a fixed number the 1st term a is called a *geometric sequence* (or *geometric progression*). r is called the *common ratio* of the geometric sequence.

1. For each given geometric sequence $\{a_n\}$, find the common ratio r and term.

Ex. 3, 6, 12, 24, ...

[Sol] $r=2$, $a_5=48$

(1) 1, 3, 9, 27, ...

[Sol] $r=3$, $a_5=81$

(2) $-\frac{1}{2}$, 1, -2, 4, ...

[Sol] $r=-2$, $a_5=-8$

(3) 4, -2, 1, $-\frac{1}{2}$, ...

[Sol] $r=-\frac{1}{2}$, $a_5=\frac{1}{4}$

(4) 5, -5, 5, -5, ...

[Sol] $r=-1$, $a_5=5$

(5) -3, -3, -3, -3, ...

[Sol] $r=1$, $a_5=-3$

2. Find the 10th term of the following geometric sequence $\{a_n\}$.

1, 2, 4, 8, 16, ...

[Sol] $a_1 = 1$

$$a_2 = 1 \cdot 2^1 = 2$$

$$a_3 = 1 \cdot 2^2 = 4$$

$$a_4 = 1 \cdot 2^3 = 8$$

$\vdots$

$$a_{10} = 1 \cdot 2^9 = 512$$

3. Find the general term of the following geometric sequence $\{a_n\}$.

5, -10, 20, -40, 80, ...

[Sol] $a_1 = 5$

$$a_2 = 5(-2)^1 = -10$$

$$a_3 = 5(-2)^2 = 20$$

$$a_4 = 5(-2)^3 = -40$$

$\vdots$

$$a_{n-1} = 5(-2)^{n-2}$$

$$a_n = 5(-2)^{n-1}$$

Find the general term of the following geometric sequence $\{a_n\}$ with 1st term a and common ratio r .

[Sol] $a_1 = a$

$$a_2 = ar$$

$$a_3 = ar^2$$

$$a_4 = ar^3$$

$\vdots$

$$a_n = ar^{n-1}$$



$a_1, a_2, a_3, \dots, a_{n-1}, a_n$
 $\times r \times r \times r \dots \times r \times r$
 The number of $\times r$ is $(n-1)$.

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

General Term of a Geometric Sequence

The general term of a geometric sequence $\{a_n\}$ with 1st term a and common ratio r is

$$a_n = ar^{n-1}$$

1. Find the general term and the 6th term of each given geometric sequence

Ex. 3, 6, 12, 24, ...

[Sol] Since the 1st term is 3 and the common ratio is 2,

$$a_n = 3 \cdot 2^{n-1}$$

$$a_6 = 3 \cdot 2^5 = 96$$

(1) 2, 6, 18, 54, ...

[Sol] Since the 1st term is 2 and the common ratio is 3,

$$a_n = 2 \cdot 3^{n-1}$$

$$a_6 = 2 \cdot 3^5 = 486$$

(2) -1, -4, -16, -64, ...

[Sol] Since the 1st term is -1 and the common ratio is 4,

$$a_n = -1 \cdot 4^{n-1} = -4^{n-1}$$

$$a_6 = -4^5 = -1024$$

(3) -2, 4, -8, 16, ...

[Sol] Since the 1st term is -2 and the common ratio is -2,

$$a_n = -2(-2)^{n-1} = (-2)^n$$

$$a_6 = (-2)^6 = 64$$

N12b

2. Find the 1st term a of the geometric sequence $\{a_n\}$ whose common ratio is $-\frac{1}{3}$ and 3rd term is 6.

$$[\text{Sol}] a_3 = a \left(-\frac{1}{3}\right)^2 = 6$$

$$\therefore a = 54$$

3. Find the common ratio r of the geometric sequence $\{a_n\}$ whose 1st term is 3 and 3rd term is 48.

$$[\text{Sol}] a_3 = 3r^2 = 48$$

$$r^2 = 16$$

$$\therefore r = \pm 4$$

4. Find the common ratio r of the geometric sequence $\{a_n\}$ whose 1st term is 5 and 4th term is 135. (r is a real number.)

$$[\text{Sol}] a_4 = 5r^3 = 135$$

$$r^3 = 27$$

$$\therefore r = 3$$

$$r^3 - 27 = 0$$

$$(r-3)(r^2+3r+9) = 0$$

Since r is a real number, $r = 3$

5. Given the geometric sequence $\{a_n\}$ whose 1st term is 16 and common ratio is $\frac{3}{2}$, find the value of n for which the n^{th} term is 81.

$$[\text{Sol}] a_n = 16 \left(\frac{3}{2}\right)^{n-1}$$

Since $a_n = 81$,

$$16 \left(\frac{3}{2}\right)^{n-1} = 81$$

$$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^4$$

$$\therefore n = 5$$

Rewriting with base $\frac{3}{2}$
(K192)

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex.

Find the general term of the geometric sequence $\{a_n\}$ whose 4th term is -24 and 6th term is -96 .

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_4 = ar^3 = -24 & \cdots \textcircled{1} \\ a_6 = ar^5 = -96 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $-24r^2 = -96$

$$r^2 = 4$$

$$\therefore r = \pm 2$$

(i) When $r = 2$, from $\textcircled{1}$, $a = -3$

$$\therefore a_n = -3 \cdot 2^{n-1}$$

(ii) When $r = -2$, from $\textcircled{1}$, $a = 3$

$$\therefore a_n = 3(-2)^{n-1}$$

From (i) and (ii), $a_n = -3 \cdot 2^{n-1}$ or $a_n = 3(-2)^{n-1}$

From $\textcircled{2}$, $ar^3 \cdot r^2 = -96$
Substituting $\textcircled{1}$ into this

1. Find the general term of the geometric sequence $\{a_n\}$ whose 3rd term is 36 and 5th term is 324 .

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_3 = ar^2 = 36 & \cdots \textcircled{1} \\ a_5 = ar^4 = 324 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $36r^2 = 324$

$$r^2 = 9$$

$$\therefore r = \pm 3$$

(i) When $r = 3$, from $\textcircled{1}$, $a = 4$

$$\therefore a_n = 4 \cdot 3^{n-1}$$

(ii) When $r = -3$, from $\textcircled{1}$, $a = 4$

$$\therefore a_n = 4(-3)^{n-1}$$

From (i) and (ii), $a_n = 4 \cdot 3^{n-1}$ or $a_n = 4(-3)^{n-1}$

N13b

2. Find the general term of the geometric sequence $\{a_n\}$ whose 2nd term is 6 and 6th term is $\frac{3}{8}$. (The common ratio is a real number.)

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_2 = ar = 6 & \dots \textcircled{1} \\ a_6 = ar^5 = \frac{3}{8} & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $6r^4 = \frac{3}{8}$

$$r^4 = \frac{1}{16}$$

$$\therefore r = \pm \frac{1}{2}$$

$$r^4 - \frac{1}{16} = 0$$

$$\left(r^2 + \frac{1}{4}\right)\left(r^2 - \frac{1}{4}\right) = 0$$

$$\left(r^2 + \frac{1}{4}\right)\left(r + \frac{1}{2}\right)\left(r - \frac{1}{2}\right) = 0$$

Since r is a real number, $r = \pm \frac{1}{2}$

(i) When $r = \frac{1}{2}$, from $\textcircled{1}$, $a = 12$

$$\therefore a_n = 12\left(\frac{1}{2}\right)^{n-1}$$

(ii) When $r = -\frac{1}{2}$, from $\textcircled{1}$, $a = -12$

$$\therefore a_n = -12\left(-\frac{1}{2}\right)^{n-1}$$

From (i) and (ii), $a_n = 12\left(\frac{1}{2}\right)^{n-1}$ or $a_n = -12\left(-\frac{1}{2}\right)^{n-1}$

3. Given the geometric sequence $\{a_n\}$ whose 4th term is $6\sqrt{3}$ and 7th term is 54, find the value of n for which the n th term is 162. (The common ratio is a real number.)

[Sol] Let a be the 1st term and r be the common ratio.

$$a_4 = ar^3 = 6\sqrt{3} \dots \textcircled{1}$$

$$a_7 = ar^6 = 54 \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $6\sqrt{3}r^3 = 54$

$$r^3 = 3\sqrt{3}$$

$$\therefore r = \sqrt{3}$$

From $\textcircled{1}$, $a = 2$

$$\therefore a_n = 2(\sqrt{3})^{n-1}$$

Since $a_n = 162$,

$$2(\sqrt{3})^{n-1} = 162$$

$$3^{\frac{1}{2}(n-1)} = 3^4$$

$$\therefore n = 9$$

$$r^3 - 3\sqrt{3} = 0$$

$$(r - \sqrt{3})(r^2 + \sqrt{3}r + 3) = 0$$

Since r is a real number, $r = \sqrt{3}$

Rewriting with base 3
(K192)

$$\frac{1}{2}(n-1) = 4$$

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 2nd term is 2 and whose sum of the first 3 terms is 7.

$$[\text{Sol}] \begin{cases} ar=2 & \dots \textcircled{1} \\ a+ar+ar^2=7 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1+r+r^2)=7$

Multiplying both sides by r ,

$$ar(1+r+r^2)=7r \quad \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $2(1+r+r^2)=7r$

$$2r^2-5r+2=0$$

$$(2r-1)(r-2)=0$$

$$\therefore r=\frac{1}{2}, 2$$

(i) When $r=\frac{1}{2}$, from $\textcircled{1}$, $a=4$

(ii) When $r=2$, from $\textcircled{1}$, $a=1$

From (i) and (ii), $a=4$, $r=\frac{1}{2}$ or $a=1$, $r=2$

1. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 2nd term is 3 and whose sum of the first 3 terms is 13.

$$[\text{Sol}] \begin{cases} ar=3 & \dots \textcircled{1} \\ a+ar+ar^2=13 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1+r+r^2)=13$

Multiplying both sides by r ,

$$ar(1+r+r^2)=13r \quad \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $3(1+r+r^2)=13r$

$$3r^2-10r+3=0$$

$$(3r-1)(r-3)=0$$

$$\therefore r=\frac{1}{3}, 3$$

(i) When $r=\frac{1}{3}$, from $\textcircled{1}$, $a=9$

(ii) When $r=3$, from $\textcircled{1}$, $a=1$

From (i) and (ii), $a=9$, $r=\frac{1}{3}$ or $a=1$, $r=3$

2. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 3rd term is 12 and whose sum of the first 3 terms is 21.

$$[\text{Sol}] \begin{cases} ar^2 = 12 & \dots \textcircled{1} \\ a + ar + ar^2 = 21 & \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{2}, a(1+r+r^2) = 21$$

Multiplying both sides by r^2 ,

$$ar^2(1+r+r^2) = 21r^2 \dots \textcircled{3}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{3}, 12(1+r+r^2) = 21r^2$$

$$3r^2 - 4r - 4 = 0$$

$$(3r+2)(r-2) = 0$$

$$\therefore r = -\frac{2}{3}, 2$$

$$(i) \text{ When } r = -\frac{2}{3}, \text{ from } \textcircled{1}, a = 27$$

$$(ii) \text{ When } r = 2, \text{ from } \textcircled{1}, a = 3$$

$$\text{From } (i) \text{ and } (ii), a = 27, r = -\frac{2}{3} \text{ or } a = 3, r = 2$$

3. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 35 and whose sum from the 3rd to the 5th term is 140.

$$[\text{Sol}] \begin{cases} a + ar + ar^2 = 35 & \dots \textcircled{1} \\ ar^2 + ar^3 + ar^4 = 140 & \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{2}, r^2(a + ar + ar^2) = 140 \dots \textcircled{3}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{3}, 35r^2 = 140$$

$$r^2 = 4$$

$$\therefore r = \pm 2$$

$$(i) \text{ When } r = 2, \text{ from } \textcircled{1}, a = 5$$

$$(ii) \text{ When } r = -2, \text{ from } \textcircled{1}, a = \frac{35}{3}$$

$$\text{From } (i) \text{ and } (ii), a = 5, r = 2 \text{ or } a = \frac{35}{3}, r = -2$$

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. Find the value of x when the sequence 2, x , 5, ... is a geometric sequence.

[Sol] Since the sequence 2, x , 5 is a geometric sequence,

$$\frac{x}{2} = \frac{5}{x}$$

$$\therefore x^2 = 10$$

$$\therefore x = \pm\sqrt{10}$$

$$\frac{2}{x} = \frac{x}{5}$$

1. Find the value of x when the sequence 3, x , 9, ... is a geometric sequence.

[Sol] Since the sequence 3, x , 9 is a geometric sequence,

$$\frac{x}{3} = \frac{9}{x}$$

$$\therefore x^2 = 27$$

$$\therefore x = \pm 3\sqrt{3}$$

2. Find the values of x and y when the sequence x , -5 , x , y , ... is a geometric sequence.

[Sol] Since the sequence x , -5 , x , y is a geometric sequence,

$$\begin{cases} \frac{-5}{x} = \frac{x}{-5} \dots \textcircled{1} \end{cases}$$

$$\begin{cases} \frac{x}{-5} = \frac{y}{x} \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1}, x^2 = 25 \dots \textcircled{3}$$

$$\therefore x = \pm 5$$

$$\text{From } \textcircled{2}, x^2 = -5y \dots \textcircled{4}$$

$$\text{From } \textcircled{3} \text{ and } \textcircled{4}, 25 = -5y$$

$$\therefore y = -5$$

Therefore, $x = 5$, $y = -5$ or $x = -5$, $y = -5$

3. Find the values of x and y when the sequence $x, y, -4$ is an arithmetic sequence and the sequence $y, x, -4$ is a geometric sequence.

[Sol] Since the sequence $x, y, -4$ is an arithmetic sequence,

$$y - x = -4 - y \quad \leftarrow \text{N4}$$

$$\therefore x = 2y + 4 \quad \dots \textcircled{1}$$

Since the sequence $y, x, -4$ is a geometric sequence,

$$\frac{x}{y} = \frac{-4}{x}$$

$$\therefore x^2 = -4y \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $(2y + 4)^2 = -4y$

$$y^2 + 5y + 4 = 0$$

$$(y + 4)(y + 1) = 0$$

$$\therefore y = -4, -1$$

(i) When $y = -4$, from $\textcircled{1}$, $x = -4$

(ii) When $y = -1$, from $\textcircled{1}$, $x = 2$

From (i) and (ii), $x = -4, y = -4$ or $x = 2, y = -1$

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

Ex. Find the sum S_4 of the geometric sequence whose 1st term is 2, common ratio is 3 and number of terms is 4.

$$\begin{aligned}
 [\text{Sol}] \quad S_4 &= 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 \\
 -) 3S_4 &= \quad 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 \\
 \hline
 S_4 - 3S_4 &= 2 \qquad \qquad \qquad -2 \cdot 3^4 \\
 \therefore -2S_4 &= 2(1 - 3^4) \\
 \therefore S_4 &= 80
 \end{aligned}$$

Multiplying both sides of S_4 by the common ratio 3

1. Find the sum S_5 of the geometric sequence whose 1st term is 3, common ratio is 2 and number of terms is 5.

$$\begin{aligned}
 [\text{Sol}] \quad S_5 &= 3 + 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 \\
 -) 2S_5 &= \quad 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 + 3 \cdot 2^5 \\
 \hline
 S_5 - 2S_5 &= 3 \qquad \qquad \qquad -3 \cdot 2^5 \\
 \therefore -S_5 &= 3(1 - 2^5) \\
 \therefore S_5 &= 93
 \end{aligned}$$

Find the sum S_n of the geometric sequence with 1st term a , common ratio r and number of terms n .

$$\begin{aligned}
 [\text{Sol}] \quad S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\
 -) rS_n &= \quad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \\
 \hline
 S_n - rS_n &= a \qquad \qquad \qquad -ar^n \\
 \therefore (1-r)S_n &= a(1-r^n)
 \end{aligned}$$

(i) When $r \neq 1$,

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}$$

(ii) When $r=1$,

$$S_n = \underbrace{a + a + a + \cdots + a}_{n \text{ terms}} = na$$

Sum of a Geometric Sequence

Let S_n be the sum of a geometric sequence with 1st term a , common ratio r and number of terms n .

$$\text{When } r \neq 1, S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

$$\text{When } r = 1, S_n = na \quad \leftarrow \quad \text{When } r = 1, S_n \text{ is the sum of } n \text{ terms of } a.$$

2. Using the formula above, find the sum of each given geometric sequence.

Ex. 1st term: -2 , common ratio: 3 ,
number of terms: 4

$$[\text{Sol}] \frac{-2(3^4-1)}{3-1} = -80$$

1st term: 3 , common ratio: 1 ,
number of terms: 10

$$[\text{Sol}] 10 \cdot 3 = 30$$

(1) 1st term: -4 , common ratio: 2 ,
number of terms: 5

$$[\text{Sol}] \frac{-4(2^5-1)}{2-1} = -124$$

(3) 1st term: -5 , common ratio: $-\frac{1}{2}$,
number of terms: 6

$$[\text{Sol}] \frac{-5 \left[1 - \left(-\frac{1}{2} \right)^6 \right]}{1 - \left(-\frac{1}{2} \right)} = -\frac{105}{32}$$

(2) 1st term: 3 , common ratio: -1 ,
number of terms: 7

$$[\text{Sol}] \frac{3[1-(-1)^7]}{1-(-1)} = 3$$

(4) 1st term: -2 , common ratio: 1 ,
number of terms: 12

$$[\text{Sol}] 12 \cdot (-2) = -24$$

Ex. S_5 , the sum of the first 5 terms of the geometric sequence $1, 3, 9, 27, \dots$

$$[\text{Sol}] S_5 = \frac{1 \cdot (3^5-1)}{3-1} = 121 \quad \leftarrow$$

1st term: 1 , common ratio: 3 ,
number of terms: 5

(5) S_7 , the sum of the first 7 terms of the geometric sequence $\frac{1}{2}, 1, 2, 4, \dots$

$$[\text{Sol}] S_7 = \frac{\frac{1}{2}(2^7-1)}{2-1} = \frac{127}{2}$$

(6) S_8 , the sum of the first 8 terms of the geometric sequence $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$

$$[\text{Sol}] S_8 = \frac{1 \cdot ((\sqrt{2})^8-1)}{\sqrt{2}-1} = \frac{15}{\sqrt{2}-1} = \frac{15(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = 15(\sqrt{2}+1)$$

Geometric Sequences

Name _____

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Time : to :

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1. Given the geometric sequence whose 1st term is 2 and common ratio is -3 , find the value of n for which the sum of the first n terms is 122.

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{2[1 - (-3)^n]}{1 - (-3)} = 122$$

$$(-3)^n = -243$$

$$(-3)^n = (-3)^5$$

$$\therefore n = 5$$

Rewriting with base -3
(K192)

2. Find the 1st term a of the geometric sequence $\{a_n\}$ whose common ratio is $\frac{1}{3}$ and whose sum of the first 4 terms is 40.

[Sol] Let S_n be the sum of the first n terms.

$$S_4 = \frac{a \left[1 - \left(\frac{1}{3} \right)^4 \right]}{1 - \frac{1}{3}} = 40$$

$$\frac{80}{81}a = \frac{80}{3}$$

$$\therefore a = 27$$

3. Find the sum S from the 7th to the 10th term of the geometric sequence $\{a_n\}$ whose 1st term is $\frac{1}{3}$ and common ratio is 2.

[Sol] Let S_n be the sum of the first n terms.

$$S_{10} = \frac{\frac{1}{3}(2^{10} - 1)}{2 - 1} = 341$$

$$S_6 = \frac{\frac{1}{3}(2^6 - 1)}{2 - 1} = 21$$

$$\therefore S = S_{10} - S_6 = 341 - 21 = 320$$

341	
$a_1 + a_2 + \dots + a_6$	$a_7 + \dots + a_{10}$
21	S

17b

Find the sum S from the 6th to the 9th term of the geometric sequence $\{a_n\}$ whose 5th term is -48 and 8th term is 384 . (The common ratio is a real number.)

[Sol] Let a be the 1st term, r be the common ratio and S_n be the sum of the first n terms.

$$\begin{cases} a_5 = ar^4 = -48 & \dots \textcircled{1} \\ a_8 = ar^7 = 384 & \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, -48r^3 = 384$$

$$r^3 = -8$$

$$\therefore r = -2$$

$$\text{From } \textcircled{1}, a = -3$$

$$\therefore S_9 = \frac{-3[1 - (-2)^9]}{1 - (-2)} = -513$$

$$S_5 = \frac{-3[1 - (-2)^5]}{1 - (-2)} = -33$$

$$\therefore S = S_9 - S_5 = -513 - (-33) = -480$$

5. Given the geometric sequence whose 1st term is 2 and common ratio is 4, find the value of n for which the sum of the first n terms becomes greater than 1000 for the first time.

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{2(4^n - 1)}{4 - 1} = \frac{2}{3}(4^n - 1)$$

$$\text{Considering } S_n > 1000,$$

$$\frac{2}{3}(4^n - 1) > 1000 \text{ when}$$

$$4^n > 1501$$

The smallest natural number n that satisfies this is $n = 6$.

Finding the smallest natural number n that satisfies $4^n > 1501$ by substituting some values into n

Geometric Sequences

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Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. Find the 1st term a and the common ratio r of the geometric sequence whose sum of the first 3 terms is 9 and whose sum of the first 6 terms is -63 . (The common ratio is a real number.)

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=9$, $S_6=6a=-63$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3-1)}{r-1} = 9 & \dots \textcircled{1} \\ S_6 = \frac{a(r^6-1)}{r-1} = -63 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $9(r^3+1) = -63$
 $r^3 = -8$

$\therefore r = -2$

From ①, $a = 3$

From (i) and (ii), $a = 3$, $r = -2$

$3a=9$, i.e. $a=3$
 Substituting this
 $6 \cdot 3 = 18 \neq -63$; $\therefore$
 this is not suitable

$$S_6 = \frac{a(r^3-1)(r^3+1)}{r-1}$$

1. Find the 1st term a and the common ratio r of the geometric sequence whose sum of the first 3 terms is 52 and whose sum of the first 6 terms is 1456. (The common ratio is a real number.)

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=52$, $S_6=6a=1456$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3-1)}{r-1} = 52 & \dots \textcircled{1} \\ S_6 = \frac{a(r^6-1)}{r-1} = 1456 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $52(r^3+1) = 1456$
 $r^3 = 27$

$\therefore r = 3$

From ①, $a = 4$

From (i) and (ii), $a = 4$, $r = 3$

2. Find the sum S from the 31st to the 60th term of the geometric sequence $\{a_n\}$ whose sum of the first 10 terms is 3 and whose sum from the 11th to the 30th term is 60. (The common ratio is a real number.)

[Sol] Let a be the 1st term, r be the common ratio and S_n be the sum of the first n terms.

- (i) When $r=1$, $S_{10}=10a=3$, $S_{30}=30a=3+60=63$ ←

No a can satisfy these conditions simultaneously.

$$\begin{array}{c} S_{30} \\ \hline \underbrace{a_1 + a_2 + \cdots + a_{10}}_3 + \underbrace{a_{11} + \cdots + a_{30}}_{60} \end{array}$$

- (ii) When $r \neq 1$,

$$\begin{cases} S_{10} = \frac{a(r^{10}-1)}{r-1} = 3 \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} S_{30} = \frac{a(r^{30}-1)}{r-1} = 63 \quad \dots \textcircled{2} \end{cases}$$

$$S_{30} = \frac{a(r^{10}-1)(r^{20}+r^{10}+1)}{r-1} = 63$$

From ① and ②, $3(r^{20}+r^{10}+1)=63$ ←

$$r^{20}+r^{10}-20=0$$

$$(r^{10}-4)(r^{10}+5)=0$$

$$r^{10}=4 \quad \dots \textcircled{3}$$

$$r^{10}+5 > 0$$

From ② and ③,

$$\begin{aligned} S_{60} &= \frac{a(r^{60}-1)}{r-1} \\ &= \frac{a(r^{30}-1)(r^{30}+1)}{r-1} \\ &= \frac{a(r^{30}-1)}{r-1} [(r^{10})^3+1] \\ &= 63(4^3+1) \end{aligned}$$

$$\begin{array}{l} \text{From ②,} \\ \frac{a(r^{30}-1)}{r-1} = 63 \end{array}$$

$$= 4095$$

From (i) and (ii), $S = S_{60} - S_{30}$

$$= 4095 - 63$$

$$= 4032$$

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. You want to save 10000 dollars in 5 years by saving x dollars at the beginning of each year. Given that the annual interest rate is 2%, find x . Use $(1.02)^5 = 1.1$ and round your answer down to the nearest 10 dollars.

[Sol] After 5 years, x dollars from the 1st year becomes $x \times (1.02)^5$ dollars

After 4 years, x dollars from the 2nd year becomes $x \times (1.02)^4$ dollars

⋮

After 1 year, x dollars from the 5th year becomes $x \times 1.02$ dollars

Let S be the amount of savings at the end of the 5th year.

$$S = 1.02x + (1.02)^2x + \cdots + (1.02)^5x \quad \leftarrow \begin{array}{l} 1^{\text{st}} \text{ term: } 1.02x, \\ \text{common ratio: } 1.02, \\ \text{number of terms: } 5 \end{array}$$

$$= \frac{1.02x[(1.02)^5 - 1]}{1.02 - 1}$$

$$= \frac{1.02(1.1 - 1)}{0.02}x$$

$$= 5.1x$$

Thus, since $S = 10000$,

$$5.1x = 10000 \quad \therefore x = 1960.7\dots$$

Rounding down to the nearest 10 dollars, 1960

Ans. 1960 dollars

1.1, 1.02, 1.02, 1.1

The method of determining the interest on the initial amount and the accumulated interest of previous periods of a deposit or loan is called the *compound interest method*. An interest calculated by the above method is called a *compound interest*.

For example, when you save 100 dollars in the bank with an annual interest rate of 2%, after 1 year, your savings becomes $100 \times 1.02 = 102$ (dollars).

After 2 years, the annual interest rate of 2% accrues on the interest of 2 dollars; therefore, savings becomes $100 \times (1.02)^2 = 104.04$ (dollars).

1. You want to save 20000 dollars in 10 years by saving x dollars at the beginning of each year. Given that the annual interest rate is 8%, find x . Use $(1.08)^{10} = 2.1$ and round your answer down to the nearest 10 dollars.

[Sol] After 10 years, x dollars from the 1st year become $x \times (1.08)^{10}$ dollars.

After 9 years, x dollars from the 2nd year become $x \times (1.08)^9$ dollars.

$\vdots$

After 1 year, x dollars from the 10th year become $x \times 1.08$ dollars.

Let S be the amount of savings at the end of the 10th year.

$$S = 1.08x + (1.08)^2x + \dots + (1.08)^{10}x$$

$$= \frac{1.08x[(1.08)^{10} - 1]}{1.08 - 1}$$

$$= \frac{1.08(2.1 - 1)}{0.08}x$$

$$= 14.85x$$

Thus, since $S = 20000$,

$$14.85x = 20000 \quad \therefore x = 1346.8\dots$$

Rounding down to the nearest 10 dollars, 1340

Ans. 1340 dollars

Geometric Sequences

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

1. Find the general term of the geometric sequence $\{a_n\}$ whose 2nd term is 12 and 4th term is 192. ➡ N

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_2 = ar = 12 & \cdots \textcircled{1} \\ a_4 = ar^3 = 192 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $12r^2 = 192$

$$r^2 = 16$$

$$\therefore r = \pm 4$$

(i) When $r = 4$, from $\textcircled{1}$, $a = 3$

$$\therefore a_n = 3 \cdot 4^{n-1}$$

(ii) When $r = -4$, from $\textcircled{1}$, $a = -3$

$$\therefore a_n = -3(-4)^{n-1}$$

From (i) and (ii), $a_n = 3 \cdot 4^{n-1}$ or $a_n = -3(-4)^{n-1}$

2. Find the 1st term a and the common ratio r of the geometric sequence whose 2nd term is -3 and whose sum of the first 3 terms is 7. ➡ N

$$[\text{Sol}] \begin{cases} ar = -3 & \cdots \textcircled{1} \\ a + ar + ar^2 = 7 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1+r+r^2) = 7$

Multiplying both sides by r ,

$$ar(1+r+r^2) = 7r \quad \cdots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $-3(1+r+r^2) = 7r$

$$3r^2 + 10r + 3 = 0$$

$$(r+3)(3r+1) = 0$$

$$\therefore r = -3, -\frac{1}{3}$$

(i) When $r = -3$, from $\textcircled{1}$, $a = 1$

(ii) When $r = -\frac{1}{3}$, from $\textcircled{1}$, $a = 9$

From (i) and (ii), $a = 1, r = -3$ or $a = 9, r = -\frac{1}{3}$

Alternative Solution

$$\begin{cases} ar = -3 & \cdots \textcircled{1} \\ \frac{a(r^3-1)}{r-1} = 7 & \cdots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{2}, a(r^2+r+1) = 7 \quad \leftarrow \frac{r^3-1}{r-1} = (r-1)(r^2+r+1)$$

Multiplying both sides by r ,

$$ar(r^2+r+1) = 7r \quad \cdots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $-3(r^2+r+1) = 7r$

$$3r^2 + 10r + 3 = 0$$

$$(r+3)(3r+1) = 0$$

$$\therefore r = -3, -\frac{1}{3}$$

(i) When $r = -3$, from $\textcircled{1}$, $a = 1$

(ii) When $r = -\frac{1}{3}$, from $\textcircled{1}$, $a = 9$

From (i) and (ii),

$$a = 1, r = -3 \text{ or } a = 9, r = -\frac{1}{3}$$

3. Given the geometric sequence whose 1st term is 6 and common ratio is -2 , find the value of n for which the sum of the first n terms is -30 . $\Rightarrow$ N17

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{6[1 - (-2)^n]}{1 - (-2)} = -30$$

$$(-2)^n = 16$$

$$(-2)^n = (-2)^4$$

$$\therefore n = 4$$

4. Find the 1st term a and the common ratio r of the geometric sequence (a_n) whose sum of the first 3 terms is 6 and whose sum of the first 6 terms is -42 . (The common ratio is a real number.) $\Rightarrow$ N18

[Sol] Let S_n be the sum of the first n terms.

(i) When $r = 1$, $S_3 = 3a = 6$, $S_6 = 6a = -42$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$S_3 = \frac{a(r^3 - 1)}{r - 1} = 6 \quad \dots \textcircled{1}$$

$$S_6 = \frac{a(r^6 - 1)}{r - 1} = -42 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $6(r^3 + 1) = -42$

$$r^3 = -8$$

$$\therefore r = -2$$

From $\textcircled{1}$, $a = 2$

From (i) and (ii), $a = 2$, $r = -2$

Alternative Solution

$$a + ar + ar^2 = 6 \quad \dots \textcircled{1}$$

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 = -42 \quad \dots \textcircled{2}$$

From $\textcircled{2}$,

$$a + ar + ar^2 + r^3(a + ar + ar^2) = -42 \quad \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $6 + 6r^3 = -42$

$$r^3 = -8$$

$$\therefore r = -2$$

From $\textcircled{1}$, $a = 2$

$$\therefore a = 2, r = -2$$

Various Sequences 1

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The summation symbol Σ is used to denote the sum of a sequence.

For example, $a_3 + a_4 + a_5 + a_6$, the sum from the 3rd to the 6th term of the sequence

$\{a_n\}$ is denoted by $\sum_{k=3}^6 a_k$.

Generally, $\sum_{k=p}^q a_k$ denotes the sum from the p^{th} to the q^{th} term of a sequence $\{a_n\}$.

(Note that other letters may be used in place of k .)

1. Let $\{a_n\}$ be the sequence

1, 3, 5, 7, 9, 11, ...

Find the following sums.

Ex.

$$\sum_{k=2}^4 a_k = a_2 + a_3 + a_4 = 3 + 5 + 7 = 15$$



$$\sum_{k=2}^4 a_k = a_2 + a_3 + a_4$$

$$(1) \sum_{k=3}^5 a_k = a_3 + a_4 + a_5 = 5 + 7 + 9 = 21$$

$$(2) \sum_{m=3}^6 a_m = a_3 + a_4 + a_5 + a_6 = 5 + 7 + 9 + 11 = 32$$

$$(3) \sum_{n=1}^5 a_n = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 5 + 7 + 9 = 25$$

$$(4) \sum_{k=1}^3 a_{k+1} = a_{1+1} + a_{2+1} + a_{3+1} = a_2 + a_3 + a_4 = 3 + 5 + 7 = 15$$

$$(5) \sum_{k=1}^3 a_{2k} = a_{2 \cdot 1} + a_{2 \cdot 2} + a_{2 \cdot 3} = a_2 + a_4 + a_6 = 3 + 7 + 11 = 21$$

$$(6) \sum_{k=1}^2 a_{2k-1} = a_{2 \cdot 1 - 1} + a_{2 \cdot 2 - 1} = a_1 + a_3 = 1 + 5 = 6$$

$$(7) \sum_{k=2}^2 a_k = a_2 = 3$$

N21b

2. Find the following sums.

Ex. $\sum_{k=2}^4 (3k-1) = (3 \cdot 2 - 1) + (3 \cdot 3 - 1) + (3 \cdot 4 - 1) = 5 + 8 + 11 = 24$

(1) $\sum_{k=3}^5 2k^2 = 2 \cdot 3^2 + 2 \cdot 4^2 + 2 \cdot 5^2 = 18 + 32 + 50 = 100$

(2) $\sum_{k=0}^2 5^k = 5^0 + 5^1 + 5^2 = 1 + 5 + 25 = 31$

(3) $\sum_{n=3}^4 (2n-1)^2 = (2 \cdot 3 - 1)^2 + (2 \cdot 4 - 1)^2 = 25 + 49 = 74$

(4) $\sum_{m=2}^3 (3^m - 2^{m-1}) = (3^2 - 2^{2-1}) + (3^3 - 2^{3-1}) = 7 + 23 = 30$

(5) $\sum_{k=4}^6 (k+n) = (\boxed{4} + n) + (\boxed{5} + n) + (\boxed{6} + n) = 3n + \boxed{15}$

(6) $\sum_{k=2}^5 (n-k) = (n-2) + (n-3) + (n-4) + (n-5) = 4n - 14$

3. Rewrite the following sums using Σ .

Ex. $2+4+6+8+10 = \sum_{k=1}^5 2k$ $\leftarrow$ Given $\{a_n\}: 2, 4, 6, \dots$, the sum of the first 5 terms of $\{a_n\}$ is $\sum_{k=1}^5 2k$.

(1) $5+10+15+20 = \sum_{k=1}^4 5k$

(2) $1+4+9+16+25+36 = \sum_{k=1}^6 k^2$

(3) $3+5+7+\dots+(2n+1) = \sum_{k=1}^n (\boxed{2k+1})$

(4) $3+12+27+\dots+3n^2 = \sum_{k=1}^n 3k^2$

(5) $3+12+27+\dots+3n^2+3(n+1)^2 = \sum_{k=1}^{n+1} \boxed{3k^2}$ $\leftarrow$

Given $\{a_n\}: 3, 12, 27, \dots$, from (4), the sum of the first n terms of $\{a_n\}$ is $\sum_{k=1}^n 3k^2$.

Adding $3(n+1)^2$ that is the $(n+1)^{\text{th}}$ term of $\{a_n\}$.

$\sum_{k=1}^n 3k^2 + 3(n+1)^2 = \sum_{k=1}^{n+1} 3k^2$

Various Sequences 1

Name _____

Date / /

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(mistakes) 0	—	1	2	3~

Since the sum S_n of the arithmetic sequence with 1st term a , last term l and number of terms n is

$$S_n = \frac{1}{2}n(a+l), \quad \leftarrow \text{N6}$$

the sum of the natural numbers from 1 to n (1, 2, 3, ..., n) becomes

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

Also, the sum of a sequence, where $a_1 = a_2 = a_3 = \cdots = a_n = c$ (c is a constant)

$$\underbrace{c+c+\cdots+c}_{n \text{ terms}} = nc$$

Rewriting the sums of these sequences using Σ , the following formulas are obtained

Summation Formulas I

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1), \quad \sum_{k=1}^n c = nc \quad (c \text{ is a constant})$$

1. Find the following sums.

Ex. $\sum_{k=1}^{10} k = \frac{1}{2} \cdot 10(10+1) = 55$ $\sum_{k=1}^{15} 2 = 15 \cdot 2 = 30$

(1) $\sum_{k=1}^{15} k = \frac{1}{2} \cdot 15(15+1) = 120$

(2) $\sum_{k=1}^{10} 5 = 10 \cdot 5 = 50$

(3) $\sum_{k=1}^{20} k - 3 \sum_{k=1}^8 2 = \frac{1}{2} \cdot 20(20+1) - 3 \cdot 8 \cdot 2 = 162$

(4) $2 \sum_{k=1}^n k - \sum_{k=1}^n 1 = 2 \cdot \frac{1}{2}n(n+1) - n \cdot 1 = n^2$

b

two sequences

$$: a_1, a_2, a_3, \dots, a_n$$

$$: b_1, b_2, b_3, \dots, b_n,$$

the following equations using Σ .

$$(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \dots + (a_n + b_n)$$

$$(a_1 + a_2 + a_3 + \dots + a_n) + (b_1 + b_2 + b_3 + \dots + b_n)$$

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \boxed{\sum_{k=1}^n b_k}$$

$$a_1 + ca_2 + ca_3 + \dots + ca_n = c(a_1 + a_2 + a_3 + \dots + a_n) \quad (c \text{ is a constant})$$

$$\sum_{k=1}^n ca_k = c \boxed{\sum_{k=1}^n a_k}$$

Answers: $\sum_{k=1}^n a_k$ $\sum_{k=1}^n b_k$

the above, the following equations are true.

Summation Properties

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k, \quad \sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k \quad (c \text{ is a constant})$$

and the following sums.

$$\sum_{k=1}^n (3k+1) = 3 \sum_{k=1}^n k + \sum_{k=1}^n 1 = 3 \cdot \frac{1}{2}n(n+1) + n = \frac{1}{2}n(3n+5)$$

$$\sum_{k=1}^n (4k+3) = 4 \sum_{k=1}^n k + \sum_{k=1}^n 3 = 4 \cdot \frac{1}{2}n(n+1) + 3n = n(2n+5)$$

$$\sum_{k=1}^n (-2k+4) = -2 \sum_{k=1}^n k + \sum_{k=1}^n 4 = -2 \cdot \frac{1}{2}n(n+1) + 4n = -n(n-3)$$

$$\begin{aligned} \sum_{k=1}^{n-1} (3k+5) &= 3 \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 5 = 3 \cdot \frac{1}{2}(n-1)[(n-1)+1] + 5(n-1) \\ &= \frac{1}{2}(n-1)(3n+10) \end{aligned}$$

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Find $\sum_{k=1}^n k^2$.[Sol] Use $(k+1)^3 - k^3 = 3k^2 + 3k + 1$. $\leftarrow (k+1)^3 = k^3 + 3k^2 + 3k + 1$ When $k=1$,

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$

When $k=2$,

$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$

When $k=3$,

$$4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$

 $\vdots$ $\vdots$ When $k=n$,

$$(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

Adding up the terms on each side from all these n equations,

$$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + \cdots + n^2) + 3(1 + 2 + \cdots + n) + (1 + 1 + \cdots + 1)$$

$$= 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + n$$

Therefore,

Rearranging into $3 \sum_{k=1}^n k^2 =$

$$3 \sum_{k=1}^n k^2 = (n+1)^3 - 1 - 3 \sum_{k=1}^n k - n$$

$$= (n+1)^3 - 3 \cdot \frac{1}{2} n(n+1) - (n+1)$$

$$= \frac{1}{2} (n+1) [2(n+1)^2 - 3n - 2]$$

$$= \frac{1}{2} n(n+1)(2n+1)$$

$$\therefore \sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1)$$

Summation Formula II

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Find the following sums.

$$\begin{aligned}\sum_{k=1}^4 (k^2 + 2k) &= \sum_{k=1}^4 k^2 + 2 \sum_{k=1}^4 k \\ &= \frac{1}{6} \cdot 4(4+1)(2 \cdot 4+1) + 2 \cdot \frac{1}{2} \cdot 4(4+1) \\ &= 50\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^n (3k^2 - k) &= 3 \sum_{k=1}^n k^2 - \sum_{k=1}^n k \\ &= 3 \cdot \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n+1)[(2n+1) - 1] \\ &= n^2(n+1)\end{aligned}$$

$$\begin{aligned}3) \quad \sum_{k=1}^n (6k^2 - 7k - 3) &= 6 \sum_{k=1}^n k^2 - 7 \sum_{k=1}^n k - \sum_{k=1}^n 3 \\ &= 6 \cdot \frac{1}{6}n(n+1)(2n+1) - 7 \cdot \frac{1}{2}n(n+1) - 3n \\ &= \frac{1}{2}n[2(n+1)(2n+1) - 7(n+1) - 6] \\ &= \frac{1}{2}n(4n^2 - n - 11)\end{aligned}$$

b

Summation Formula III

$$\sum_{k=1}^n k^3 = \left[\frac{1}{2}n(n+1) \right]^2$$

Find the following sums.

$$\begin{aligned}\sum_{k=1}^6 (k^3 + k) &= \sum_{k=1}^6 k^3 + \sum_{k=1}^6 k \\ &= \left[\frac{1}{2} \cdot 6(6+1) \right]^2 + \frac{1}{2} \cdot 6(6+1) \\ &= 462\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^n (2k^3 - 3k) &= 2\sum_{k=1}^n k^3 - 3\sum_{k=1}^n k \\ &= 2\left[\frac{1}{2}n(n+1) \right]^2 - 3 \cdot \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n+1)[n(n+1) - 3] \\ &= \frac{1}{2}n(n+1)(n^2 + n - 3)\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^n (4k^3 - 6k^2 + 3) &= 4\sum_{k=1}^n k^3 - 6\sum_{k=1}^n k^2 + \sum_{k=1}^n 3 \\ &= 4\left[\frac{1}{2}n(n+1) \right]^2 - 6 \cdot \frac{1}{6}n(n+1)(2n+1) + 3n \\ &= n[n(n+1)^2 - (n+1)(2n+1) + 3] \\ &= n(n^3 - 2n + 2)\end{aligned}$$

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Since the sum S_n of the geometric sequence with 1st term a , common ratio r ($r \neq 1$) and number of terms n is

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \quad \leftarrow \text{NI6}$$

when $r \neq 1$, the sum of the geometric sequence $a, ar, ar^2, \dots, ar^{n-1}$ becomes

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Rewriting the sum of the sequence using Σ , the following formula is obtained

Summation Formula IV

$$\sum_{k=1}^n ar^{k-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \quad (r \neq 1)$$

Find the following sums.

Ex.

$$\sum_{k=1}^4 2 \cdot 3^{k-1} = \frac{2(3^4-1)}{3-1} = 80$$

$$(1) \quad \sum_{k=1}^6 5 \cdot 2^{k-1} = \frac{5(2^6-1)}{2-1} = 315$$

$$(2) \quad \sum_{k=1}^3 (-3)^{k-1} = \frac{1-(-3)^3}{1-(-3)} = 7$$

$$(3) \quad \sum_{k=1}^n 3 \cdot 4^{k-1} = \frac{3(4^n-1)}{4-1} = 4^n - 1$$

$$(4) \quad \sum_{k=1}^n 6(-5)^{k-1} = \frac{6[1-(-5)^n]}{1-(-5)} = 1 - (-5)^n$$

N25b

$$(5) \sum_{k=1}^n 4^k = \sum_{k=1}^n 4 \cdot 4^{k-1} = \frac{4(4^n - 1)}{4 - 1} = \frac{4^{n+1} - 4}{3}$$

$$(6) \sum_{k=1}^n 6^k = \sum_{k=1}^n 6 \cdot 6^{k-1} = \frac{6(6^n - 1)}{6 - 1} = \frac{6^{n+1} - 6}{5}$$

$$(7) \sum_{k=1}^n 3(-2)^k = \sum_{k=1}^n [-6(-2)^{k-1}] = \frac{-6[1 - (-2)^n]}{1 - (-2)} = -2 - (-2)^{n+1}$$

$$\begin{aligned} (8) \sum_{k=1}^n (4 \cdot 3^k - 2^{k+1}) &= \sum_{k=1}^n 12 \cdot 3^{k-1} - \sum_{k=1}^n 4 \cdot 2^{k-1} \\ &= \frac{12(3^n - 1)}{3 - 1} - \frac{4(2^n - 1)}{2 - 1} \\ &= (2 \cdot 3^{n+1} - 6) - (2^{n+2} - 4) \\ &= 2 \cdot 3^{n+1} - 2^{n+2} - 2 \\ &= [2(3^{n+1} - 2^{n+2} - 1)] \end{aligned}$$

$$\begin{aligned} (9) \sum_{k=1}^{n-1} (5^k + 3^{k+1}) &= \sum_{k=1}^{n-1} 5 \cdot 5^{k-1} + \sum_{k=1}^{n-1} 9 \cdot 3^{k-1} \\ &= \frac{5(5^{n-1} - 1)}{5 - 1} + \frac{9(3^{n-1} - 1)}{3 - 1} \\ &= \left(\frac{5^n}{4} - \frac{5}{4}\right) + \left(\frac{3^{n+1}}{2} - \frac{9}{2}\right) \\ &= \frac{5^n}{4} + \frac{3^{n+1}}{2} - \frac{23}{4} \\ &= \left[\frac{1}{4}(5^n + 2 \cdot 3^{n+1} - 23)\right] \end{aligned}$$

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Find the following sums.

$$(1) \sum_{k=1}^n (2k+3^k)$$

$$= 2 \sum_{k=1}^n k + \sum_{k=1}^n 3 \cdot 3^{k-1}$$

$$= 2 \cdot \frac{1}{2} n(n+1) + \frac{3(3^n-1)}{3-1}$$

$$= n(n+1) + \frac{3^{n+1}-3}{2}$$

$$(2) \sum_{k=1}^n [3k^2 + (-5)^k]$$

$$= 3 \sum_{k=1}^n k^2 + \sum_{k=1}^n [-5(-5)^{k-1}]$$

$$= 3 \cdot \frac{1}{6} n(n+1)(2n+1) + \frac{-5[1-(-5)^n]}{1-(-5)}$$

$$= \frac{1}{2} n(n+1)(2n+1) - \frac{5+(-5)^{n+1}}{6}$$

$$(3) \sum_{k=1}^n (6k^2 + 2k + 4^{k+1})$$

$$= 6 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k + \sum_{k=1}^n 16 \cdot 4^{k-1}$$

$$= 6 \cdot \frac{1}{6} n(n+1)(2n+1) + 2 \cdot \frac{1}{2} n(n+1) + \frac{16(4^n-1)}{4-1}$$

$$= n(n+1)[(2n+1)+1] + \frac{4^{n+2}-16}{3}$$

$$= 2n(n+1)^2 + \frac{4^{n+2}-16}{3}$$

N26b

$$(4) \sum_{k=1}^{2n} (2k^3 + 3k^2 + 6^{k-1})$$

$$= 2 \sum_{k=1}^{2n} k^3 + 3 \sum_{k=1}^{2n} k^2 + \sum_{k=1}^{2n} 6^{k-1}$$

$$= 2 \left[\frac{1}{2} \cdot 2n(2n+1) \right]^2 + 3 \cdot \frac{1}{6} \cdot 2n(2n+1)(2 \cdot 2n+1) + \frac{6^{2n}-1}{6-1}$$

$$= n(2n+1)[2n(2n+1) + (4n+1)] + \frac{6^{2n}-1}{5}$$

$$= n(2n+1)(4n^2+6n+1) + \frac{6^{2n}-1}{5}$$

$$(5) \quad \sum_{k=1}^{n+1} \left(4k^{\frac{3}{2}} + 3^k \right) \left(4k^{\frac{3}{2}} - 3^k \right)$$

$$= \sum_{k=1}^{n+1} (16k^3 - 9^k)$$

$$= 16 \sum_{k=1}^{n+1} k^3 - \sum_{k=1}^{n+1} 9 \cdot 9^{k-1}$$

$$= 16 \left\{ \frac{1}{2} (n+1) [(n+1)+1] \right\}^2 - \frac{9(9^{n+1}-1)}{9-1}$$

$$= 4(n+1)^2(n+2)^2 - \frac{9^{n+2}-9}{8}$$

$$(6) \quad \sum_{k=1}^{n-1} (-3k + 2^{k+1})(3k + 2^{k+1})$$

$$= \sum_{k=1}^{n-1} (-9k^2 + 4^{k+1})$$

$$= -9 \sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} 16 \cdot 4^{k-1} \frac{(1-4^n)}{1-4} = \dots$$

$$-9 \cdot \frac{1}{6}(n-1)[(n-1)+1][2(n-1)+1] + \frac{16(4^{n-1}-1)}{4-1}$$

$$-\frac{3}{2}n(n-1)(2n-1) + \frac{4^{n-1}-16}{3}$$

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Find the following sums.

Ex. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2)$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $k(k+2)$,

$$\begin{aligned}
 \sum_{k=1}^n k(k+2) &= \sum_{k=1}^n (k^2 + 2k) \\
 &= \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\
 &= \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) \\
 &= \frac{1}{6}n(n+1)[(2n+1)+6] \\
 &= \frac{1}{6}n(n+1)(2n+7)
 \end{aligned}$$

(1) $2 \cdot 2 + 4 \cdot 5 + 6 \cdot 8 + \cdots + 2n(3n-1)$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $2k(3k-1)$,

$$\begin{aligned}
 \sum_{k=1}^n 2k(3k-1) &= \sum_{k=1}^n (6k^2 - 2k) \\
 &= 6 \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k \\
 &= 6 \cdot \frac{1}{6}n(n+1)(2n+1) - 2 \cdot \frac{1}{2}n(n+1) \\
 &= n(n+1)[(2n+1)-1] \\
 &= 2n^2(n+1)
 \end{aligned}$$

$$2) 1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \dots + n^2(n+1)$$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $k^2(k+1)$,

$$\begin{aligned} \sum_{k=1}^n k^2(k+1) &= \sum_{k=1}^n (k^3 + k^2) \\ &= \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\ &= \left[\frac{1}{2}n(n+1) \right]^2 + \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{12}n(n+1)[3n(n+1) + 2(2n+1)] \\ &= \frac{1}{12}n(n+1)(n+2)(3n+1) \end{aligned}$$

$$(3) 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $(2k-1)^3$,

$$\begin{aligned} \sum_{k=1}^n (2k-1)^3 &= \sum_{k=1}^n (8k^3 - 12k^2 + 6k - 1) \\ &= 8 \sum_{k=1}^n k^3 - 12 \sum_{k=1}^n k^2 + 6 \sum_{k=1}^n k - \sum_{k=1}^n 1 \\ &= 8 \left[\frac{1}{2}n(n+1) \right]^2 - 12 \cdot \frac{1}{6}n(n+1)(2n+1) + 6 \cdot \frac{1}{2}n(n+1) - n \\ &= n[2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1] \\ &= n(2n^2 - 1) \end{aligned}$$

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Find the sum S_n of the first n terms of each given sequence.**Ex.** $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \dots$ [Sol] The k^{th} term is $k(k+1) = k^2 + k$.

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n (k^2 + k) = \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\
 &= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\
 &= \frac{1}{6}n(n+1)[(2n+1)+3] \\
 &= \frac{1}{3}n(n+1)(n+2)
 \end{aligned}$$

(1) $1 \cdot 4, 2 \cdot 7, 3 \cdot 10, 4 \cdot 13, \dots$ [Sol] The k^{th} term is $k(3k+1) = 3k^2 + k$.

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n (3k^2 + k) = 3 \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\
 &= 3 \cdot \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\
 &= \frac{1}{2}n(n+1)[(2n+1)+1] \\
 &= n(n+1)^2
 \end{aligned}$$

(2) $1 \cdot 2 \cdot 3, 2 \cdot 3 \cdot 4, 3 \cdot 4 \cdot 5, 4 \cdot 5 \cdot 6, \dots$ [Sol] The k^{th} term is $k(k+1)(k+2) = k^3 + 3k^2 + 2k$.

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n (k^3 + 3k^2 + 2k) \\
 &= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\
 &= \left[\frac{1}{2}n(n+1) \right]^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) \\
 &= \frac{1}{4}n(n+1)[n(n+1) + 2(2n+1) + 4] \\
 &= \frac{1}{4}n(n+1)(n+2)(n+3)
 \end{aligned}$$

b

 $1+2, 1+2+3, 1+2+3+4, \dots$

The k^{th} term is $1+2+3+\dots+k = \frac{1}{2}k(k+1) = \frac{1}{2}(k^2+k)$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n \frac{1}{2}(k^2+k) = \frac{1}{2} \left(\sum_{k=1}^n k^2 + \sum_{k=1}^n k \right) && 1+2+3+\dots+n = \sum_{k=1}^n k \\ &= \frac{1}{2} \left[\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \right] \\ &= \frac{1}{12}n(n+1)[(2n+1)+3] \\ &= \frac{1}{6}n(n+1)(n+2)\end{aligned}$$

 $1^2, 1^2+2^2, 1^2+2^2+3^2, 1^2+2^2+3^2+4^2, \dots$

The k^{th} term is

$$1^2+2^2+3^2+\dots+k^2 = \frac{1}{6}k(k+1)(2k+1) = \frac{1}{6}(2k^3+3k^2+k). \quad \leftarrow$$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n \frac{1}{6}(2k^3+3k^2+k) && 1^2+2^2+3^2+\dots+n^2 = \sum_{k=1}^n k^2 \\ &= \frac{1}{6} \left(2\sum_{k=1}^n k^3 + 3\sum_{k=1}^n k^2 + \sum_{k=1}^n k \right) \\ &= \frac{1}{6} \left\{ 2 \left[\frac{1}{2}n(n+1) \right]^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \right\} \\ &= \frac{1}{12}n(n+1)[n(n+1) + (2n+1) + 1] \\ &= \frac{1}{12}n(n+1)^2(n+2)\end{aligned}$$

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1. Find the sum of the first 30 terms of the sequence $\{a_n = \alpha + \beta n + \gamma n^2\}$ whose term is 3, 2nd term is 5 and whose sum of the first 3 terms is 17.

$$[\text{Sol}] \begin{cases} a_1 = 3 & \dots \textcircled{1} \\ a_2 = 5 & \dots \textcircled{2} \\ a_1 + a_2 + a_3 = 17 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1} \sim \textcircled{3}$,

$$a_3 = 9$$

$$\therefore \begin{cases} a_1 = \alpha + \beta + \gamma = 3 & \dots \textcircled{4} \\ a_2 = \alpha + 2\beta + 4\gamma = 5 & \dots \textcircled{5} \\ a_3 = \alpha + 3\beta + 9\gamma = 9 & \dots \textcircled{6} \end{cases}$$

From $\textcircled{4} \sim \textcircled{6}$,

$$\alpha = 3, \beta = -1, \gamma = 1$$

$$\therefore a_n = 3 - n + n^2$$

Therefore, the sum is

$$\begin{aligned} \sum_{k=1}^{30} a_k &= \sum_{k=1}^{30} (3 - k + k^2) \\ &= \sum_{k=1}^{30} 3 - \sum_{k=1}^{30} k + \sum_{k=1}^{30} k^2 \\ &= 30 \cdot 3 - \frac{1}{2} \cdot 30(30+1) + \frac{1}{6} \cdot 30(30+1)(2 \cdot 30+1) \\ &= 9080 \end{aligned}$$

N29b

2. Given $S_n = \sum_{k=1}^n k$ ($n=1, 2, \dots$), find the following sums.

(1) $T_n = \sum_{k=1}^n S_k$

[Sol] $S_n = \sum_{k=1}^n k = \frac{1}{2}n(n+1) = \frac{1}{2}(n^2+n)$

$$\therefore T_n = \sum_{k=1}^n \frac{1}{2}(k^2+k) \quad \leftarrow S_k = \frac{1}{2}(k^2+k)$$

$$= \frac{1}{2} \left(\sum_{k=1}^n k^2 + \sum_{k=1}^n k \right)$$

$$= \frac{1}{2} \left[\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \right]$$

$$= \frac{1}{12}n(n+1)[(2n+1)+3]$$

$$= \frac{1}{6}n(n+1)(n+2)$$

(2) $U_n = \sum_{k=1}^n T_k$

[Sol] From (1), $T_n = \frac{1}{6}(n^3+3n^2+2n)$

$$\therefore U_n = \sum_{k=1}^n \frac{1}{6}(k^3+3k^2+2k) \quad \leftarrow T_k = \frac{1}{6}(k^3+3k^2+2k)$$

$$= \frac{1}{6} \left(\sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \right)$$

$$= \frac{1}{6} \left\{ \left[\frac{1}{2}n(n+1) \right]^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) \right\}$$

$$= \frac{1}{24}n(n+1)[n(n+1)+2(2n+1)+4]$$

$$= \frac{1}{24}n(n+1)(n+2)(n+3)$$

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1. Find the following sums.

$$\begin{aligned}
 (1) \quad \sum_{k=1}^n (6k^2 - 2k + 3) &= 6 \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k + \sum_{k=1}^n 3 \\
 &= 6 \cdot \frac{1}{6} n(n+1)(2n+1) - 2 \cdot \frac{1}{2} n(n+1) + 3n \\
 &= n[(n+1)(2n+1) - (n+1) + 3] \\
 &= n(2n^2 + 2n + 3)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \sum_{k=1}^n (k^3 + 2k) &= \sum_{k=1}^n k^3 + 2 \sum_{k=1}^n k \\
 &= \left[\frac{1}{2} n(n+1) \right]^2 + 2 \cdot \frac{1}{2} n(n+1) \\
 &= \frac{1}{4} n(n+1)[n(n+1) + 4] \\
 &= \frac{1}{4} n(n+1)(n^2 + n + 4)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \sum_{k=1}^n (2^k + 4 \cdot 5^{k-1}) &= \sum_{k=1}^n 2 \cdot 2^{k-1} + \sum_{k=1}^n 4 \cdot 5^{k-1} \\
 &= \frac{2(2^n - 1)}{2 - 1} + \frac{4(5^n - 1)}{5 - 1} \\
 &= (2^{n+1} - 2) + (5^n - 1) \\
 &= 2^{n+1} + 5^n - 3
 \end{aligned}$$

N30b

$$(4) \quad 2^2 \cdot 2 + 4^2 \cdot 3 + 6^2 \cdot 4 + \cdots + (2n)^2(n+1)$$

⇒ N27

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $(2k)^2(k+1)$,

$$\begin{aligned} \sum_{k=1}^n (2k)^2(k+1) &= \sum_{k=1}^n 4(k^3 + k^2) \\ &= 4 \left(\sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \right) \\ &= 4 \left\{ \left[\frac{1}{2}n(n+1) \right]^2 + \frac{1}{6}n(n+1)(2n+1) \right\} \\ &= \frac{1}{3}n(n+1)[3n(n+1) + 2(2n+1)] \\ &= \frac{1}{3}n(n+1)(n+2)(3n+1) \end{aligned}$$

2. Find the sum S_n of the first n terms of the following sequence.

⇒ N28

$$1^2 \cdot 3, 2^2 \cdot 5, 3^2 \cdot 7, 4^2 \cdot 9, \dots$$

[Sol] The k^{th} term is $k^2(2k+1) = 2k^3 + k^2$.

$$\begin{aligned} \therefore S_n &= \sum_{k=1}^n (2k^3 + k^2) = 2 \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\ &= 2 \left[\frac{1}{2}n(n+1) \right]^2 + \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{6}n(n+1)[3n(n+1) + (2n+1)] \\ &= \frac{1}{6}n(n+1)(3n^2 + 5n + 1) \end{aligned}$$

Various Sequences 2

Name _____

Date / /

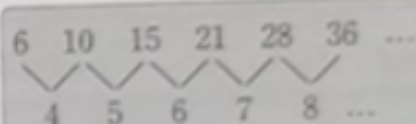
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1. Find the sequence by writing out the difference between each two consecutive terms in each given sequence.

Ex. 6, 10, 15, 21, 28, 36, ...

[Sol] 4, 5, 6, 7, 8, ...



(1) 1, 2, 4, 7, 11, 16, ...

[Sol] 1, 2, 3, 4, 5, ...

(2) 3, 6, 11, 18, 27, 38, ...

[Sol] 3, 5, 7, 9, 11, ...

(3) 1, 4, 11, 22, 37, 56, ...

[Sol] 3, 7, 11, 15, 19, ...

(4) 18, 17, 14, 9, 2, -7, ...

[Sol] -1, -3, -5, -7, -9, ...

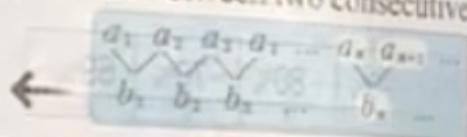
(5) 2, 4, 6, 8, 10, 12, ...

[Sol] 2, 2, 2, 2, 2, ...

N31b

The term which is derived by taking the difference between two consecutive terms in the sequence $\{a_n\}$ is

$$b_n = a_{n+1} - a_n \quad (n=1, 2, 3, \dots)$$



Generally, the sequence $\{b_n\}$ is called the *sequence of differences* of $\{a_n\}$.

2. Given that the sequence $\{a_n\}$ is 1, 3, 6, 10, 15, ..., its sequence of differences is 2, 3, 4, 5, Using this, find the general term of $\{a_n\}$. ($n \geq 2$)

[Sol]

$$\begin{aligned} 3-1 &= 2 \\ 6-3 &= 3 \\ 10-6 &= 4 \\ &\vdots \\ a_n - a_{n-1} &= n \end{aligned}$$

Adding up the terms on each side from all $(n-1)$ equalities,

$$\begin{aligned} a_n - \boxed{1} &= 2+3+4+\dots+n \\ \therefore a_n &= \boxed{1} + (2+3+4+\dots+n) \\ &= \boxed{\frac{1}{2}n(n+1)} \end{aligned}$$

3. Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$. Using $\{b_n\}$, find the general term of $\{a_n\}$. ($n \geq 2$)

[Sol]

$$\begin{aligned} a_2 - a_1 &= b_1 \\ a_3 - a_2 &= b_2 \\ a_4 - a_3 &= b_3 \\ &\vdots \\ a_n - a_{n-1} &= b_{n-1} \end{aligned}$$

Adding up the terms on each side from all $(n-1)$ equalities,

$$\begin{aligned} a_n - \boxed{a_1} &= b_1 + b_2 + b_3 + \dots + b_{n-1} \\ \therefore a_n &= \boxed{a_1} + (b_1 + b_2 + b_3 + \dots + b_{n-1}) \\ &= \boxed{a_1} + \sum_{k=1}^{n-1} b_k \end{aligned}$$

Various Sequences 2

Name _____

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Sequence of Differences and General Term

Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$.

$$\text{When } n \geq 2, a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

Find the general term of each given sequence $\{a_n\}$.**Ex.** 2, 6, 12, 20, 30, 42, ...[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$

$$4, 6, 8, 10, 12, \dots$$

$$\therefore b_n = 4 + (n-1) \cdot 2 = 2n + 2$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} (2k + 2)$$

$$= 2 + 2 \cdot \frac{1}{2} (n-1)n + 2(n-1)$$

$$= n(n+1) \dots \textcircled{1}$$

Since $a_1 = 2$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = n(n+1)$$

$\{b_n\}$ is the arithn
sequence with 1st
and common diffe

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1), \sum_{k=1}^n 2k = n(n+1)$$

Substituting $n = 1$ i
 $a_1 = 2$ and it coinci
with the 1st term of
given sequence.

(1) 2, 10, 24, 44, 70, 102, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$8, 14, 20, 26, 32, \dots$$

$$\therefore b_n = 8 + (n-1) \cdot 6 = 6n + 2$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} (6k + 2)$$

$$= 2 + 6 \cdot \frac{1}{2} (n-1)n + 2(n-1)$$

$$= n(3n-1) \dots \textcircled{1}$$

Since $a_1 = 2$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = n(3n-1)$$

N32b

(2) 10, 10, 9, 7, 4, 0, ...

[Sol] Let (b_n) be the sequence of differences of (a_n) . Then, (b_n) is

0, -1, -2, -3, -4, ...

$$\therefore b_n = 0 + (n-1) \cdot (-1) = -n+1$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} (-k+1)$$

$$= 10 - \frac{1}{2}(n-1)n + (n-1)$$

$$= -\frac{1}{2}(n+3)(n-6) \dots \textcircled{1}$$

Since $a_1 = 10$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = -\frac{1}{2}(n+3)(n-6)$$

(3) $1, \frac{9}{2}, \frac{20}{2}, \frac{35}{2}, \frac{54}{2}, \frac{77}{2}, \dots$

[Sol] Let (b_n) be the sequence of differences of (a_n) . Then, (b_n) is

$\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, \dots$

$$\therefore b_n = \frac{7}{2} + (n-1) \cdot 2 = 2n + \frac{3}{2}$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} \left(2k + \frac{3}{2}\right)$$

$$= 1 + 2 \cdot \frac{1}{2}(n-1)n + \frac{3}{2}(n-1)$$

$$= \frac{1}{2}(n+1)(2n-1) \dots \textcircled{1}$$

$a_1 = 1$, $\textcircled{1}$ is also true when $n=1$.

$$a_n = \frac{1}{2}(n+1)(2n-1)$$

Various Sequences 2

Name _____

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Find the general term of each given sequence $\{a_n\}$.

- (1) 1, 4, 10, 22, 46, 94, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

3, 6, 12, 24, 48, ...

$\therefore b_n = 3 \cdot 2^{n-1}$

 $\{b_n\}$ is the geometric sequence with 1st term 3 and common ratio 2.When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 3 \cdot 2^{k-1}$$

$$= 1 + \frac{3(2^{n-1} - 1)}{2 - 1}$$

$$= 3 \cdot 2^{n-1} - 2 \quad \cdots \textcircled{1}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = 3 \cdot 2^{n-1} - 2$$

- (2) 1, 3, 7, 15, 31, 63, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

2, 4, 8, 16, 32, ...

$\therefore b_n = 2 \cdot 2^{n-1} = 2^n$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 2^k$$

$$= 1 + \sum_{k=1}^{n-1} 2 \cdot 2^{k-1}$$

$$= 1 + \frac{2(2^{n-1} - 1)}{2 - 1}$$

$$= 2^n - 1 \quad \cdots \textcircled{1}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = 2^n - 1$$

N33b

(3) 3, 4, 1, 10, -17, 64, ...

[Sol] Let (b_n) be the sequence of differences of (a_n) . Then, (b_n) is

1, -3, 9, -27, 81, ...

$$\therefore b_n = 1 \cdot (-3)^{n-1} = (-3)^{n-1}$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (-3)^{k-1} \\ &= 3 + \frac{1 - (-3)^n}{1 - (-3)} \\ &= \frac{13 - (-3)^n}{4} \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1 = 3$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = \frac{13 - (-3)^n}{4}$$

(4) $\frac{1}{6}, \frac{1}{9}, \frac{1}{14}, \frac{1}{21}, \frac{1}{30}, \frac{1}{41}, \dots$

(Let $b_n = \frac{1}{a_n}$ and (c_n) be the sequence of differences of (b_n) .)

[Sol] Let $b_n = \frac{1}{a_n}$. (b_n) is 6, 9, 14, 21, 30, 41, ...

Let (c_n) be the sequence of differences of (b_n) . Then, (c_n) is

3, 5, 7, 9, 11, ...

$$\therefore c_n = 3 + (n-1) \cdot 2 = 2n+1$$

When $n \geq 2$,

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} (2k+1) \\ &= 6 + 2 \cdot \frac{1}{2} (n-1)n + (n-1) \\ &= n^2 + 5 \quad \dots \textcircled{1} \end{aligned}$$

Since $b_1 = 6$, $\textcircled{1}$ is also true when $n = 1$.

$$b_n = n^2 + 5$$

$$a_n = \frac{1}{b_n} = \frac{1}{n^2 + 5}$$

Various Sequences 2

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Time : to :

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Given that S_n is the sum of the first n terms of the sequence $\{a_n\}$, fill in following blanks and find the relationship between S_n and a_n .

[Sol] When $n \geq 2$, $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \cdots \textcircled{1}$

$$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1} \cdots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$, $S_n - S_{n-1} = a_n$

Also, $S_1 = a_1$

From the above, the following is true.

Sum of a Sequence and General Term

Let S_n be the sum of the first n terms of the sequence $\{a_n\}$.

The 1st term a_1 is $a_1 = S_1$.

When $n \geq 2$, $a_n = S_n - S_{n-1}$

Find the general term of each given sequence $\{a_n\}$ whose sum of the first n terms is S_n .

Ex. $S_n = 2n^2 - 3n$

[Sol] The 1st term a_1 is $a_1 = S_1 = 2 \cdot 1^2 - 3 \cdot 1 = -1$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (2n^2 - 3n) - [2(n-1)^2 - 3(n-1)] \\ &= 4n - 5 \cdots \textcircled{1} \end{aligned}$$

Since $a_1 = -1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = 4n - 5$$

N34b

(1) $S_n = n^2 + 4n$

[Sol] The 1st term a_1 is $a_1 = S_1 = 1^2 + 4 \cdot 1 = 5$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (n^2 + 4n) - [(n-1)^2 + 4(n-1)] \\ &= 2n + 3 \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1 = 5$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 2n + 3$

(2) $S_n = 2^n - 1$

[Sol] The 1st term a_1 is $a_1 = S_1 = 2^1 - 1 = 1$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (2^n - 1) - (2^{n-1} - 1) \\ &= 2^{n-1} \quad \dots \textcircled{1} \end{aligned}$$

$$\begin{aligned} 2^n - 2^{n-1} &= 2 \cdot 2^{n-1} - 2^{n-1} \\ &= (2-1)2^{n-1} \end{aligned}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 2^{n-1}$

Various Sequences 2

Name _____

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Find the sum S for each given question.

Ex. $S = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 3^2 + 8 \cdot 3^3 + \dots + 2n \cdot 3^{n-1} \dots \textcircled{1}$

[Sol] Multiplying both sides of $\textcircled{1}$ by 3,Since $\textcircled{1}$ is

$$S = 2 \cdot 1 + 4 \cdot 3 + \dots + 2(n-1) \cdot 3^{n-2} + 2n \cdot 3^{n-1}$$

$$3S = 2 \cdot 3 + 4 \cdot 3^2 + 6 \cdot 3^3 + \dots + 2(n-1) \cdot 3^{n-1} + 2n \cdot 3^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-2S = (2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \dots + 2 \cdot 3^{n-1}) - 2n \cdot 3^n$$

$$= \frac{2(3^n - 1)}{3 - 1} - 2n \cdot 3^n$$

$$= -(2n - 1) \cdot 3^n - 1$$

$$\therefore S = \frac{(2n - 1) \cdot 3^n + 1}{2}$$

$$\begin{array}{r} S = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 3^2 + 8 \cdot 3^3 + \dots + 2n \cdot 3^{n-1} \\ - 3S = 2 \cdot 3 + 4 \cdot 3^2 + 6 \cdot 3^3 + \dots + 2(n-1) \cdot 3^{n-1} + 2n \cdot 3^n \\ \hline S - 3S = 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \dots + 2 \cdot 3^{n-1} - 2n \cdot 3^n \end{array}$$

(1) $S = 1 \cdot 1 + 2 \cdot 5 + 3 \cdot 5^2 + 4 \cdot 5^3 + \dots + n \cdot 5^{n-1} \dots \textcircled{1}$

[Sol] Multiplying both sides of $\textcircled{1}$ by 5,

$$5S = 1 \cdot 5 + 2 \cdot 5^2 + 3 \cdot 5^3 + \dots + (n-1) \cdot 5^{n-1} + n \cdot 5^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-4S = (1 + 5 + 5^2 + 5^3 + \dots + 5^{n-1}) - n \cdot 5^n$$

$$= \frac{5^n - 1}{5 - 1} - n \cdot 5^n$$

$$= -\frac{(4n - 1) \cdot 5^n + 1}{4}$$

$$\therefore S = \frac{(4n - 1) \cdot 5^n + 1}{16}$$

N35b

$$(2) \quad S = n \cdot 1 + (n-1) \cdot 3 + (n-2) \cdot 3^2 + (n-3) \cdot 3^3 + \dots + 2 \cdot 3^{n-2} + 1 \cdot 3^{n-1} \dots \textcircled{1}$$

[Sol] Multiplying both sides of $\textcircled{1}$ by 3,

$$3S = n \cdot 3 + (n-1) \cdot 3^2 + (n-2) \cdot 3^3 + \dots + 3 \cdot 3^{n-2} + 2 \cdot 3^{n-1} + 3^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-2S = n - (3 + 3^2 + 3^3 + \dots + 3^{n-2} + 3^{n-1} + 3^n)$$

$$= n - \frac{3(3^n - 1)}{3 - 1}$$

$$= \frac{2n - 3^{n+1} + 3}{2}$$

$$\therefore S = \frac{2n - 3^{n+1} + 3}{4}$$

$$(3) \quad S = 4 \cdot 1 + 7 \cdot 2 + 10 \cdot 2^2 + 13 \cdot 2^3 + \dots + (3n+1) \cdot 2^{n-1} \dots \textcircled{1}$$

[Sol] Multiplying both sides of $\textcircled{1}$ by 2,

$$2S = 4 \cdot 2 + 7 \cdot 2^2 + 10 \cdot 2^3 + \dots + (3n-2) \cdot 2^{n-1} + (3n+1) \cdot 2^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-S = 4 + (3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \dots + 3 \cdot 2^{n-1}) - (3n+1) \cdot 2^n$$

$$= 4 + \frac{6(2^{n-1}-1)}{2-1} - (3n+1) \cdot 2^n$$

$$= -(3n-2) \cdot 2^n - 2$$

$$\therefore S = (3n-2) \cdot 2^n + 2$$

Alternative Solution

Multiplying both sides of ① by 2,

$$2S = 4 \cdot 2 + 7 \cdot 2^2 + 10 \cdot 2^2 + \dots + (3n-2) \cdot 2^{n-1} + (3n+1) \cdot 2^n \quad \text{--- ②}$$

From ① - ②,

$$-S = 1 + (3 + 3 \cdot 2 + 3 \cdot 2^2 + \dots + 3 \cdot 2^{n-1}) - (3n+1) \cdot 2^n$$

$$= 1 + \frac{3(2^n - 1)}{2 - 1} - (3n+1) \cdot 2^n$$

$$= -(3n-2) \cdot 2^n - 2$$

$$\therefore S = (3n-2) \cdot 2^n + 2$$

Various Sequences 2

Name _____

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Time : to :

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Find the sum S for each given question.

Ex. $S = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}}$

[Sol] $\frac{1}{\sqrt{k}+\sqrt{k+1}} = \frac{\sqrt{k}-\sqrt{k+1}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k}-\sqrt{k+1})}$
 $= -\sqrt{k}+\sqrt{k+1}$

$$\begin{array}{r} -\sqrt{1}+\sqrt{2} \\ -\sqrt{2}+\sqrt{3} \\ -\sqrt{3}+\sqrt{4} \\ \vdots \\ -\sqrt{n}+\sqrt{n+1} \\ \hline -1+\sqrt{n+1} \end{array}$$

Therefore,

$$S = (-\sqrt{1}+\sqrt{2}) + (-\sqrt{2}+\sqrt{3}) + (-\sqrt{3}+\sqrt{4}) + \cdots + (-\sqrt{n}+\sqrt{n+1})$$

$$= -1 + \sqrt{n+1}$$

(1) $S = \frac{2}{\sqrt{1}+\sqrt{3}} + \frac{2}{\sqrt{3}+\sqrt{5}} + \frac{2}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{2}{\sqrt{2n-1}+\sqrt{2n+1}}$

[Sol] $\frac{2}{\sqrt{2k-1}+\sqrt{2k+1}} = \frac{2(\sqrt{2k-1}-\sqrt{2k+1})}{(\sqrt{2k-1}+\sqrt{2k+1})(\sqrt{2k-1}-\sqrt{2k+1})}$
 $= -\sqrt{2k-1}+\sqrt{2k+1}$

Therefore,

$$S = (-\sqrt{1}+\sqrt{3}) + (-\sqrt{3}+\sqrt{5}) + (-\sqrt{5}+\sqrt{7}) + \cdots + (-\sqrt{2n-1}+\sqrt{2n+1})$$

$$= -1 + \sqrt{2n+1}$$

N36b

$$(2) \quad S = \frac{1}{\sqrt{2} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{8}} + \frac{1}{\sqrt{8} + \sqrt{11}} + \dots + \frac{1}{\sqrt{3n-1} + \sqrt{3n+2}}$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{\sqrt{3k-1} + \sqrt{3k+2}} &= \frac{\sqrt{3k-1} - \sqrt{3k+2}}{(\sqrt{3k-1} + \sqrt{3k+2})(\sqrt{3k-1} - \sqrt{3k+2})} \\ &= -\frac{1}{3}(\sqrt{3k-1} - \sqrt{3k+2}) \end{aligned}$$

Therefore,

$$\begin{aligned} S &= -\frac{1}{3}(\sqrt{2} - \sqrt{5}) - \frac{1}{3}(\sqrt{5} - \sqrt{8}) - \frac{1}{3}(\sqrt{8} - \sqrt{11}) - \dots - \frac{1}{3}(\sqrt{3n-1} - \sqrt{3n+2}) \\ &= -\frac{1}{3}(\sqrt{2} - \sqrt{3n+2}) \end{aligned}$$

$$(3) \quad S = \frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{2} + \sqrt{4}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \dots + \frac{1}{\sqrt{n} + \sqrt{n+2}}$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{\sqrt{k} + \sqrt{k+2}} &= \frac{\sqrt{k} - \sqrt{k+2}}{(\sqrt{k} + \sqrt{k+2})(\sqrt{k} - \sqrt{k+2})} \\ &= -\frac{1}{2}(\sqrt{k} - \sqrt{k+2}) \end{aligned}$$

Therefore,

$$\begin{aligned} S &= -\frac{1}{2}(\sqrt{1} - \sqrt{3}) - \frac{1}{2}(\sqrt{2} - \sqrt{4}) - \frac{1}{2}(\sqrt{3} - \sqrt{5}) - \dots \\ &\quad \dots - \frac{1}{2}(\sqrt{n-2} - \sqrt{n}) - \frac{1}{2}(\sqrt{n-1} - \sqrt{n+1}) - \frac{1}{2}(\sqrt{n} - \sqrt{n+2}) \\ &= -\frac{1}{2}(1 + \sqrt{2} - \sqrt{n+1} - \sqrt{n+2}) \end{aligned}$$

$$\begin{aligned} &-\frac{1}{2}(\sqrt{1} - \sqrt{3}) \\ &-\frac{1}{2}(\sqrt{2} - \sqrt{4}) \\ &-\frac{1}{2}(\sqrt{3} - \sqrt{5}) \\ &\vdots \\ &-\frac{1}{2}(\sqrt{n-2} - \sqrt{n}) \\ &-\frac{1}{2}(\sqrt{n-1} - \sqrt{n+1}) \\ &-\frac{1}{2}(\sqrt{n} - \sqrt{n+2}) \\ &= -\frac{1}{2}(1 + \sqrt{2} - \sqrt{n+1} - \sqrt{n+2}) \end{aligned}$$

Various Sequences 2

Name _____

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Find the sum S for each given question.**Ex.**

$$S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

[Sol] Let $\frac{1}{k(k+1)} = \frac{a}{k} - \frac{b}{k+1}$.

$$1 = a(k+1) - bk$$

$$= (a-b)k + a$$

$$\therefore \begin{cases} a-b=0 \\ a=1 \end{cases}$$

$$\therefore a=1, b=1$$

Multiplying both sides by $k(k+1)$

Coefficient Comparison Method (J181)

Therefore,

$$\begin{aligned} S &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

$$(1) \quad S = \frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \cdots + \frac{2}{(2n-1)(2n+1)}$$

[Sol] Let $\frac{2}{(2k-1)(2k+1)} = \frac{a}{2k-1} - \frac{b}{2k+1}$.

$$2 = a(2k+1) - b(2k-1)$$

$$= 2(a-b)k + (a+b)$$

$$\therefore \begin{cases} 2(a-b)=0 \\ a+b=2 \end{cases}$$

$$\therefore a=1, b=1$$

Therefore,

$$\begin{aligned} S &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= 1 - \frac{1}{2n+1} = \frac{2n}{2n+1} \end{aligned}$$

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \dots + \frac{1}{2n(2n+2)}$$

$$\text{et } \frac{1}{2k(2k+2)} = \frac{a}{2k} - \frac{b}{2k+2}.$$

$$1 = a(2k+2) - b \cdot 2k$$

$$= 2(a-b)k + 2a$$

$$\therefore \begin{cases} 2(a-b)=0 \\ 2a=1 \end{cases}$$

$$\therefore a = \frac{1}{2}, b = \frac{1}{2}$$

Therefore,

$$S = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{4} - \frac{1}{6} \right) + \frac{1}{2} \left(\frac{1}{6} - \frac{1}{8} \right) + \dots + \frac{1}{2} \left(\frac{1}{2n} - \frac{1}{2n+2} \right)$$

$$= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2n+2} \right) = \frac{n}{4(n+1)}$$

$$= \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{n(n+2)}$$

$$\text{et } \frac{1}{k(k+2)} = \frac{a}{k} - \frac{b}{k+2}.$$

$$1 = a(k+2) - bk$$

$$= (a-b)k + 2a$$

$$\therefore \begin{cases} a-b=0 \\ 2a=1 \end{cases}$$

$$\therefore a = \frac{1}{2}, b = \frac{1}{2}$$

Therefore,

$$S = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \dots$$

$$+ \frac{1}{2} \left(\frac{1}{n-2} - \frac{1}{n} \right) + \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{n(3n+5)}{4(n+1)(n+2)}$$

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) \\ & \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) \\ & \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) \\ & \vdots \\ & \frac{1}{2} \left(\frac{1}{n-2} - \frac{1}{n} \right) \\ & \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\ & + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ & \hline & \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \end{aligned}$$

Various Sequences 2

Name _____

Date ____/____/____

Time ____ : ____ : ____

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. Partitioning the sequence of positive odd numbers into groups as shown below, assume that the n^{th} group contains n terms.

1 | 3, 5 | 7, 9, 11 | 13, 15, 17, 19 | 21, ...
 1st group 2nd group 3rd group 4th group

(1) The m^{th} term is the first odd number in the 10th group. Find m .

[Sol] The number of terms from the 1st to the 9th group is:

1 | 3, 5 | 7, 9, 11 | ... | 9th group |
 1 term 2 terms 3 terms 9 terms

$$1+2+3+\cdots+9=\sum_{k=1}^9 k=\frac{1}{2}\cdot 9(9+1)=45$$

$$\therefore m = 46$$

Since the last term in the 9th group is the 45th term, the first term in the 10th group is $45+1=46$.

(2) Find the first odd number x in the n^{th} group.

[Sol] The n^{th} group contains n terms.

When $n \geq 2$, the number of terms from the 1st to the $(n-1)^{\text{th}}$ group

$$1+2+3+\cdots+(n-1)=\sum_{k=1}^{n-1} k=\frac{1}{2}(n-1)n$$

Therefore, the first odd number in the n^{th} group is the $\left[\frac{1}{2}(n-1)n + 1\right]$ term.

$$\therefore x = 2\left[\frac{1}{2}(n-1)n + 1\right] - 1 = n^2 - n + 1 \quad \cdots \textcircled{1}$$

Since $x=1$ when $n=1$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore x = n^2 - n + 1$$

Since the sequence of positive odd numbers is the arithmetic sequence with 1st term common difference 2, the n^{th} term is $1+(n-1)\cdot 2=2n-1$.

(3) Find the sum S of all odd numbers in the n^{th} group.

[Sol] Since S is the sum of the arithmetic sequence with 1st term $n^2 - n + 1$ common difference 2 and number of terms n ,

$$S = \frac{1}{2}n[2(n^2 - n + 1) + (n-1)\cdot 2] = n^3$$

N38b

Partitioning the sequence of natural numbers into groups as shown below, assume that the n^{th} group contains $2n$ terms.

1. 2 | 3, 4, 5, 6 | 7, 8, 9, 10, 11, 12 | 13, ...
 1st group 2nd group 3rd group

(1) The m^{th} term is the first natural number in the 8th group. Find m .

[Sol] The number of terms from the 1st to the 7th group is:

$$2 + 4 + 6 + \dots + 14 = \sum_{k=1}^7 2k = 2 \cdot \frac{1}{2} \cdot 7(7+1) = 56$$

$$\therefore m = 57$$

1. 2 | 3, 4, 5, 6 | ... | 7th group |
 2 terms 4 terms 14 terms

(2) Find the first natural number x in the n^{th} group.

[Sol] The n^{th} group contains $2n$ terms.

When $n \geq 2$, the number of terms from the 1st to the $(n-1)^{\text{th}}$ group is:

$$2 + 4 + 6 + \dots + 2(n-1) = \sum_{k=1}^{n-1} 2k = 2 \cdot \frac{1}{2} (n-1)n = (n-1)n$$

Therefore, the first natural number in the n^{th} group is the $[(n-1)n + 1]^{\text{th}}$ term.

$$\therefore x = (n-1)n + 1 = n^2 - n + 1 \quad \dots \textcircled{1}$$

The k^{th} term of the sequence of natural numbers is k .

Since $x = 1$ when $n = 1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore x = n^2 - n + 1$$

(3) Find the sum S of all natural numbers in the n^{th} group.

[Sol] Since S is the sum of the arithmetic sequence with 1st term $n^2 - n + 1$, common difference 1 and number of terms $2n$,

$$S = \frac{1}{2} \cdot 2n [2(n^2 - n + 1) + (2n - 1) \cdot 1] = n(2n^2 + 1)$$

Various Sequences 2

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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1. Find the general term of the following sequence.

6, 24, 60, 120, 210, 336, 504, ...

[Sol] Let $\{a_n\}$ be the given sequence, $\{b_n\}$ be the sequence of differences $\{a_n\}$, and $\{c_n\}$ be the sequence of the differences of $\{b_n\}$.

$$\{a_n\}: 6, 24, 60, 120, 210, 336, 504, \dots$$

$$\{b_n\}: 18, 36, 60, 90, 126, 168, \dots$$

$$\{c_n\}: 18, 24, 30, 36, 42, \dots$$

$$\therefore c_n = 18 + (n-1) \cdot 6 = 6n + 12$$

When $n \geq 2$,

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} (6k + 12) \\ &= 18 + 6 \cdot \frac{1}{2} (n-1)n + 12(n-1) \\ &= 3n^2 + 9n + 6 \quad \dots \textcircled{1} \end{aligned}$$

Since $b_1 = 18$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore b_n = 3n^2 + 9n + 6$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (3k^2 + 9k + 6) \\ &= 6 + 3 \cdot \frac{1}{6} (n-1)n(2n-1) + 9 \cdot \frac{1}{2} (n-1)n + 6(n-1) \\ &= n(n+1)(n+2) \quad \dots \textcircled{2} \end{aligned}$$

Since $a_1 = 6$, $\textcircled{2}$ is also true when $n = 1$.

$$\therefore a_n = n(n+1)(n+2)$$

2. Given the sequence $1, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \dots$, the m^{th} term is $\frac{5}{11}$. Find m . Also, find the 48th term.

[Sol] Partitioning the sequence into groups by the common denominator,

$$1 \left| \frac{1}{2}, \frac{2}{2} \right| \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \left| \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \right| \dots$$

Therefore, $\frac{5}{11}$ is the 5th term in the 11th group.

Since the n^{th} group contains n terms,
the number of terms from the 1st to
the 10th group is:

The 11th group contains the fractions
with denominator 11.

$$\dots \left| \frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \dots, \frac{11}{11} \right| \dots$$

$1^{\text{st}} \qquad \qquad \qquad 5^{\text{th}}$

$$1+2+3+\dots+10 = \sum_{k=1}^{10} k = \frac{1}{2} \cdot 10(10+1) = 55$$

$$\therefore m = 60$$

Since $\frac{5}{11}$ is the 5th term in the 11th group,
 $55+5=60$

Also, let the 48th term be in the n^{th} group.

$$\frac{1}{2}(n-1)n < 48 \leq \frac{1}{2}n(n+1)$$

$$(n-1)n < 96 \leq n(n+1)$$

The natural number n that satisfies this
is $n = 10$.

The number of terms from the
1st to the $(n-1)^{\text{th}}$ group is:

$$1+2+3+\dots+(n-1) = \sum_{k=1}^{n-1} k$$

The number of terms from the
1st to the n^{th} group is:

$$1+2+3+\dots+n = \sum_{k=1}^n k$$

Finding the natural number n that satisfies
 $(n-1)n < 96 \leq n(n+1)$ by substituting some values into n

The number of terms from the 1st to the 9th group is:

$$1+2+3+\dots+9 = \sum_{k=1}^9 k = \frac{1}{2} \cdot 9(9+1) = 45$$

Therefore, the 48th term is $\frac{3}{10}$.

Since $48-45=3$, the 48th term
is the 3rd term in the 10th group.

Various Sequences 2

Name _____

Date / /

Time : to :

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1. Find the general term of the following sequence $\{a_n\}$. ➡ N

7, 19, 37, 61, 91, 127, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

12, 18, 24, 30, 36, ...

$$\therefore b_n = 12 + (n-1) \cdot 6 = 6n + 6$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (6k+6) \\ &= 7 + 6 \cdot \frac{1}{2} (n-1)n + 6(n-1) \\ &= 3n^2 + 3n + 1 \quad \cdots \textcircled{1} \end{aligned}$$

Since $a_1 = 7$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = 3n^2 + 3n + 1$$

2. Find the general term of the sequence $\{a_n\}$ whose sum of the first n terms is S_n . ➡ 1

$$S_n = 5n - n^2$$

[Sol] The 1st term a_1 is $a_1 = S_1 = 5 \cdot 1 - 1^2 = 4$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (5n - n^2) - [5(n-1) - (n-1)^2] \\ &= -2n + 6 \quad \cdots \textcircled{1} \end{aligned}$$

Since $a_1 = 4$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = -2n + 6$$

3. Find the sum S for each given question.

$$(1) S = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 3^2 + 4 \cdot 3^3 + \dots + n \cdot 3^{n-1} \quad \text{--- ①}$$

► N35

[Sol] Multiplying both sides of ① by 3,

$$3S = 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \dots + (n-1) \cdot 3^{n-1} + n \cdot 3^n \quad \text{--- ②}$$

From ① - ②,

$$-2S = (1 + 3 + 3^2 + 3^3 + \dots + 3^{n-1}) - n \cdot 3^n$$

$$= \frac{3^n - 1}{3 - 1} - n \cdot 3^n$$

$$= -\frac{(2n-1) \cdot 3^n + 1}{2}$$

$$\therefore S = \frac{(2n-1) \cdot 3^n + 1}{4}$$

$$(2) S = \frac{3}{1 \cdot 4} + \frac{3}{4 \cdot 7} + \frac{3}{7 \cdot 10} + \dots + \frac{3}{(3n-2)(3n+1)} \quad \text{--- ①} \quad \text{► N37}$$

[Sol] Let $\frac{3}{(3k-2)(3k+1)} = \frac{a}{3k-2} - \frac{b}{3k+1}$

$$3 = a(3k+1) - b(3k-2)$$

$$= 3(a-b)k + (a+2b)$$

$$\therefore \begin{cases} 3(a-b) = 0 \\ a+2b = 3 \end{cases}$$

$$\therefore a = 1, b = 1$$

Therefore,

$$S = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{10} \right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right)$$

$$= 1 - \frac{1}{3n+1} = \frac{3n}{3n+1}$$

Recurrence Relations

Name _____

Date / /

Time : to :

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Given that the sequence $\{a_n\}$ satisfies the following two conditions:

(i) $a_1 = 2$

(ii) $a_{n+1} = 3a_n - 1 \quad (n = 1, 2, 3, \dots)$,

fill in the following blanks.

[Sol] $a_2 = 3a_1 - 1 = 3 \cdot \boxed{2} - 1 = \boxed{5}$

$a_3 = 3a_2 - 1 = 3 \cdot \boxed{5} - 1 = \boxed{14}$

$a_4 = 3a_3 - 1 = 3 \cdot \boxed{14} - 1 = \boxed{41}$

ANS: 2, 5, 14, 41

From the above, each term is successively determined from a_1 to a_2 , a_3 , a_4 one after another, then one sequence $\{a_n\}$ will be defined. A condition such (ii), where the expression defines the relationship between two or more consecutive terms in a sequence, is called a **recurrence relation**.

Find the 4th term of each given sequence $\{a_n\}$ defined by the following condition

Ex. $a_1 = 4, a_{n+1} = 2a_n - 3$

[Sol] $a_2 = 2a_1 - 3 = 5$

$a_3 = 2a_2 - 3 = 7$

$\therefore a_4 = 2a_3 - 3 = 11$

(1) $a_1 = 1, a_{n+1} = 3a_n + 2$

[Sol] $a_2 = 3a_1 + 2 = 5$

$a_3 = 3a_2 + 2 = 17$

$\therefore a_4 = 3a_3 + 2 = 53$

N41b

$$(2) \quad a_1 = -2, \quad a_{n+1} = a_n - 5$$

$$[\text{Sol}] \quad a_2 = a_1 - 5 = -7$$

$$a_3 = a_2 - 5 = -12$$

$$\therefore a_4 = a_3 - 5 = -17$$

$$(3) \quad a_1 = 2, \quad a_{n+1} = 3a_n$$

$$[\text{Sol}] \quad a_2 = 3a_1 = 6$$

$$a_3 = 3a_2 = 18$$

$$\therefore a_4 = 3a_3 = 54$$

$$(4) \quad a_1 = 1, \quad a_{n+1} = 2 - \frac{1}{a_n}$$

$$[\text{Sol}] \quad a_2 = 2 - \frac{1}{a_1} = 1$$

$$a_3 = 2 - \frac{1}{a_2} = 1$$

$$\therefore a_4 = 2 - \frac{1}{a_3} = 1$$

$$(5) \quad a_1 = 5, \quad a_{n+1} = a_n + n$$

$$[\text{Sol}] \quad a_2 = a_1 + 1 = 6$$

$$a_3 = a_2 + 2 = 8$$

$$\therefore a_4 = a_3 + 3 = 11$$

$$=1. \quad a_2=2. \quad a_{n+2}=a_{n+1}+a_n$$

$$a_3=a_2+a_1=3$$

$$\therefore a_4=a_3+a_2=5$$

Recurrence Relations

Name _____

Date / /

Time : to :

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 1, a_{n+1} = a_n + 5$

[Sol] $a_1 = 1$

$$a_2 = a_1 + 5 = 6 \quad \begin{array}{c} 1, 6, 11, \dots \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad +5 \quad +5 \end{array}$$

$$a_3 = a_2 + 5 = 11$$

$$\vdots$$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1st term 1 and common difference 5.

$$\therefore a_n = 1 + (n-1) \cdot 5$$

$$= 5n - 4$$

The general term of an arithmetic sequence is $a_n = a + (n-1)d$.

$a_1 = 3, a_{n+1} = 2a_n$

[Sol] $a_1 = 3$

$$a_2 = 2a_1 = 6 \quad \begin{array}{c} 3, 6, 12, \dots \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad \times 2 \quad \times 2 \end{array}$$

$$a_3 = 2a_2 = 12$$

$$\vdots$$

Therefore, $\{a_n\}$ is the geometric sequence with 1st term 3 and common ratio 2.

$$\therefore a_n = 3 \cdot 2^{n-1}$$

The general term of a geometric sequence is $a_n = ar^{n-1}$.

(1) $a_1 = 4, a_{n+1} = a_n + 8$

[Sol] $a_1 = 4$

$$a_2 = a_1 + 8 = 12$$

$$a_3 = a_2 + 8 = 20$$

$$\vdots$$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1st term 4 and common difference 8.

$$\therefore a_n = 4 + (n-1) \cdot 8$$

$$= 8n - 4$$

(2) $a_1 = a, a_{n+1} = a_n + d$

[Sol] $a_1 = a$

$$a_2 = a_1 + d = a + d$$

$$a_3 = a_2 + d = a + 2d$$

$$\vdots$$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1st term a and common difference d .

$$\therefore a_n = a + (n-1)d$$

(3) $a_1 = 5, a_{n+1} = 4a_n$

[Sol] $a_1 = 5$

$$a_2 = 4a_1 = 20$$

$$a_3 = 4a_2 = 80$$

$$\vdots$$

Therefore, $\{a_n\}$ is the geometric sequence with 1st term 5 and common ratio 4.

$$\therefore a_n = 5 \cdot 4^{n-1}$$

(4) $a_1 = a, a_{n+1} = ra_n$

[Sol] $a_1 = a$

$$a_2 = ra_1 = ar$$

$$a_3 = ra_2 = ar^2$$

$$\vdots$$

Therefore, $\{a_n\}$ is the geometric sequence with 1st term a and common ratio r .

$$\therefore a_n = ar^{n-1}$$

N42b

Based on the results on side a, the following statements are true.
 The sequence $\{a_n\}$ defined by $a_1 = a$, $a_{n+1} = a_n + d$ is an arithmetic sequence with 1st term a and common difference d .
 The sequence $\{a_n\}$ defined by $a_1 = a$, $a_{n+1} = ra_n$ is a geometric sequence with 1st term a and common ratio r .

Ex.

$$a_1 = 2, a_{n+1} = a_n + 3$$

$$\begin{aligned} \text{[Sol]} \quad a_n &= 2 + (n-1) \cdot 3 \\ &= 3n - 1 \end{aligned}$$

$\{a_n\}$ is the arithmetic sequence with 1st term 2 and common difference 3.

$$a_1 = 3, a_{n+1} = 7a_n$$

$$\text{[Sol]} \quad a_n = 3 \cdot 7^{n-1}$$

$\{a_n\}$ is the geometric sequence with 1st term 3 and common ratio 7.

$$(5) \quad a_1 = -7, a_{n+1} = a_n + 5$$

$$\begin{aligned} \text{[Sol]} \quad a_n &= -7 + (n-1) \cdot 5 \\ &= 5n - 12 \end{aligned}$$

$$(8) \quad a_1 = 2, a_{n+1} = 3a_n$$

$$\text{[Sol]} \quad a_n = 2 \cdot 3^{n-1}$$

$$(6) \quad a_1 = 3, a_{n+1} = a_n - 7$$

$$\begin{aligned} \text{[Sol]} \quad a_n &= 3 + (n-1) \cdot (-7) \\ &= -7n + 10 \end{aligned}$$

$$(9) \quad a_1 = -5, a_{n+1} = -2a_n$$

$$\text{[Sol]} \quad a_n = -5(-2)^{n-1}$$

$$(7) \quad a_1 = -4, a_{n+1} = a_n - 9$$

$$\begin{aligned} \text{[Sol]} \quad a_n &= -4 + (n-1) \cdot (-9) \\ &= -9n + 5 \end{aligned}$$

$$(10) \quad a_1 = -7, a_{n+1} = 7a_n$$

$$\begin{aligned} \text{[Sol]} \quad a_n &= -7 \cdot 7^{n-1} \\ &= -7^n \end{aligned}$$

Recurrence Relations

Name _____

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 1, a_{n+1} = a_n + 3n$

[Sol] $a_{n+1} - a_n = 3n$

Since the general term of the sequence of differences of $\{a_n\}$ is $3n$,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 3k = 1 + 3 \cdot \frac{1}{2}(n-1)n$$

$$= \frac{1}{2}(3n^2 - 3n + 2) \quad \cdots \textcircled{1}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = \frac{1}{2}(3n^2 - 3n + 2)$$

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$.

$$\{a_n\}: a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$$

$$\{b_n\}: 3, 6, \dots, 3n, \dots$$

Therefore, the general term of $\{b_n\}$ is $3n$.

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

(N32)

(1) $a_1 = 3, a_{n+1} = a_n + 6n$

[Sol] $a_{n+1} - a_n = 6n$

Since the general term of the sequence of differences of $\{a_n\}$ is $6n$,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 6k = 3 + 6 \cdot \frac{1}{2}(n-1)n$$

$$= 3(n^2 - n + 1) \quad \cdots \textcircled{1}$$

Since $a_1 = 3$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = 3(n^2 - n + 1)$$

(2) $a_1 = 2, a_{n+1} = a_n + n^2$

[Sol] $a_{n+1} - a_n = n^2$

Since the general term of the sequence of differences of $\{a_n\}$ is n^2 ,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} k^2 = 2 + \frac{1}{6}(n-1)n(2n-1)$$

$$= \frac{1}{6}(2n^3 - 3n^2 + n + 12) \quad \cdots \textcircled{1}$$

Since $a_1 = 2$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = \frac{1}{6}(2n^3 - 3n^2 + n + 12)$$

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

N43b

(3) $a_1 = 4, a_{n+1} = a_n + 4n^3$

[Sol] $a_{n+1} - a_n = 4n^3$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n^3$,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 4k^3 = 4 + 4 \left[\frac{1}{2} (n-1)n \right]^2 \quad \leftarrow \quad \sum_{k=1}^n k^3 = \left[\frac{1}{2} n(n+1) \right]^2$$

$$= n^4 - 2n^3 + n^2 + 4 \quad \dots \textcircled{1}$$

Since $a_1 = 4$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = n^4 - 2n^3 + n^2 + 4$

(4) $a_1 = 3, a_{n+1} = a_n + 2^{n-1}$

[Sol] $a_{n+1} - a_n = 2^{n-1}$

Since the general term of the sequence of differences of $\{a_n\}$ is 2^{n-1} ,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 2^{k-1} = 3 + \frac{2^{n-1} - 1}{2 - 1} \quad \leftarrow \quad \sum_{k=1}^n ar^{k-1} = \frac{a(r^n - 1)}{r - 1} \quad (r \neq 1)$$

$$= 2^{n-1} + 2 \quad \dots \textcircled{1}$$

Since $a_1 = 3$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 2^{n-1} + 2$

(5) $a_1 = 4, a_{n+1} = a_n + 5^n$

[Sol] $a_{n+1} - a_n = 5^n$

Since the general term of the sequence of differences of $\{a_n\}$ is 5^n ,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 5^k = 4 + \sum_{k=1}^{n-1} 5 \cdot 5^{k-1}$$

$$= 4 + \frac{5(5^{n-1} - 1)}{5 - 1}$$

$$= \frac{5^n + 11}{4} \quad \dots \textcircled{1}$$

Since $a_1 = 4$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = \frac{5^n + 11}{4}$

4b

$$a_1 = 4, a_{n+1} = 2a_n + 1$$

[S] Since $a_{n+1} = 2a_n + 1$, $a_{n+1} + 1 = 2(a_n + 1)$

$$\text{Let } b_n = a_n + 1$$

$$\begin{cases} b_{n+1} = 2b_n & \dots \textcircled{1} \\ b_1 = a_1 + 1 = 5 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n = 5 \cdot 2^{n-1}$

$$\therefore a_n = b_n - 1 = 5 \cdot 2^{n-1} - 1$$

$$a_1 = 1, a_{n+1} = 2a_n + 3$$

[S] Since $a_{n+1} = 2a_n + 3$, $a_{n+1} + 3 = 2(a_n + 3)$

$$\text{Let } b_n = a_n + 3$$

$$\begin{cases} b_{n+1} = 2b_n & \dots \textcircled{1} \\ b_1 = a_1 + 3 = 4 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n = 4 \cdot 2^{n-1} = 2^{n+1}$

$$\therefore a_n = b_n - 3 = 2^{n+1} - 3$$

$$(3) a_1 = 1, 3a_{n+1} - 2a_n = 3$$

[Sol] Since $3a_{n+1} - 2a_n = 3$, $a_{n+1} = \frac{2}{3}a_n + 1$; therefore,

$$a_{n+1} - 3 = \frac{2}{3}(a_n - 3)$$

$$\text{Let } b_n = a_n - 3$$

$$\begin{cases} b_{n+1} = \frac{2}{3}b_n & \dots \textcircled{1} \\ b_1 = a_1 - 3 = -2 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n = -2 \left(\frac{2}{3}\right)^{n-1}$

$$\therefore a_n = b_n + 3 = -2 \left(\frac{2}{3}\right)^{n-1} + 3$$

← Replacing both a_{n+1} and a_n with x ,
 $x = 2x + 1$
 $\therefore x = -1$

← Replacing both a_{n+1} and a_n with x ,
 $x = 2x + 3$
 $\therefore x = -3$

← Replacing both a_{n+1} and a_n with x ,
 $x = \frac{2}{3}x + 1$
 $\therefore x = 3$

Note Summary

A recurrence relation $a_{n+1} = pa_n + q$ can be rearranged into $a_{n+1} - x = p(a_n - x)$ by using x , which satisfies $x = px + q$. If $b_n = a_n - x$, the sequence $\{b_n\}$ is a geometric sequence. Therefore, the general term a_n can be determined.

Recurrence Relations

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex.

$$a_1 = 1, a_{n+1} = \frac{a_n}{4a_n + 2}$$

[Sol] Since $a_{n+1} = \frac{a_n}{4a_n + 2}$, $\frac{1}{a_{n+1}} = \frac{4a_n + 2}{a_n} = \frac{2}{a_n} + 4$

Let $b_n = \frac{1}{a_n}$. Since $b_{n+1} = 2b_n + 4$,

$$\begin{cases} b_{n+1} + 4 = 2(b_n + 4) \cdots \textcircled{1} \\ b_1 + 4 = \frac{1}{a_1} + 4 = 5 \cdots \textcircled{2} \end{cases}$$

From ① and ②, $b_n + 4 = 5 \cdot 2^{n-1}$
 $b_n = 5 \cdot 2^{n-1} - 4$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{5 \cdot 2^{n-1} - 4}$$

Substituting $\frac{1}{a_n} = b_n$ and $\frac{1}{a_{n+1}} = b_{n+1}$ into

$$\frac{1}{a_{n+1}} = \frac{2}{a_n} + 4$$

Replacing both b_{n+1} and b_n with x , $x = 2x + 4$
 $\therefore x = -4$

$\{b_n + 4\}$ is the geometric sequence with 1st term $b_1 + 4 = 5$ and common ratio 2.

(1) $a_1 = 2, a_{n+1} = \frac{a_n}{2a_n + 3}$

[Sol] Since $a_{n+1} = \frac{a_n}{2a_n + 3}$, $\frac{1}{a_{n+1}} = \frac{2a_n + 3}{a_n} = \frac{3}{a_n} + 2$

Let $b_n = \frac{1}{a_n}$. Since $b_{n+1} = 3b_n + 2$,

$$\begin{cases} b_{n+1} + 1 = 3(b_n + 1) \cdots \textcircled{1} \\ b_1 + 1 = \frac{1}{a_1} + 1 = \frac{3}{2} \cdots \textcircled{2} \end{cases}$$

From ① and ②, $b_n + 1 = \frac{3}{2} \cdot 3^{n-1} = \frac{3^n}{2}$

$$b_n = \frac{3^n}{2} - 1$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{\frac{3^n}{2} - 1} = \frac{2}{3^n - 2}$$

N45b

$$(2) \quad a_1 = 1, \quad a_{n+1} = \frac{2a_n}{a_n + 10}$$

$$[\text{Sol}] \quad \text{Since } a_{n+1} = \frac{2a_n}{a_n + 10}, \quad \frac{1}{a_{n+1}} = \frac{a_n + 10}{2a_n} = \frac{5}{a_n} + \frac{1}{2}$$

$$\text{Let } b_n = \frac{1}{a_n}. \quad \text{Since } b_{n+1} = 5b_n + \frac{1}{2},$$

$$\begin{cases} b_{n+1} + \frac{1}{8} = 5\left(b_n + \frac{1}{8}\right) \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 + \frac{1}{8} = \frac{1}{a_1} + \frac{1}{8} = \frac{9}{8} \quad \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, \quad b_n + \frac{1}{8} = \frac{9}{8} \cdot 5^{n-1}$$

$$b_n = \frac{9}{8} \cdot 5^{n-1} - \frac{1}{8}$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{\frac{9}{8} \cdot 5^{n-1} - \frac{1}{8}} = \frac{8}{9 \cdot 5^{n-1} - 1}$$

$$(3) \quad a_1 = 3, \quad a_{n+1} = \frac{2a_n}{3a_n + 5}$$

$$[\text{Sol}] \quad \text{Since } a_{n+1} = \frac{2a_n}{3a_n + 5}, \quad \frac{1}{a_{n+1}} = \frac{3a_n + 5}{2a_n} = \frac{5}{2a_n} + \frac{3}{2}$$

$$\text{Let } b_n = \frac{1}{a_n}. \quad \text{Since } b_{n+1} = \frac{5}{2}b_n + \frac{3}{2},$$

$$\begin{cases} b_{n+1} + 1 = \frac{5}{2}(b_n + 1) \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 + 1 = \frac{1}{a_1} + 1 = \frac{4}{3} \quad \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, \quad b_n + 1 = \frac{4}{3} \left(\frac{5}{2}\right)^{n-1}$$

$$b_n = \frac{4}{3} \left(\frac{5}{2}\right)^{n-1} - 1$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{\frac{4}{3} \left(\frac{5}{2}\right)^{n-1} - 1} = \frac{3 \cdot 2^{n-1}}{4 \cdot 5^{n-1} - 3 \cdot 2^{n-1}}$$

Recurrence Relations

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 1, a_{n+1} = 2a_n - 3^n$

[Sol] Dividing both sides of $a_{n+1} = 2a_n - 3^n$ by 3^{n+1} ,

$$\frac{a_{n+1}}{3^{n+1}} = \frac{2}{3} \cdot \frac{a_n}{3^n} - \frac{1}{3}$$

Let $b_n = \frac{a_n}{3^n}$. Since $b_{n+1} = \frac{2}{3}b_n - \frac{1}{3}$,

$$\begin{cases} b_{n+1} + 1 = \frac{2}{3}(b_n + 1) \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 + 1 = \frac{a_1}{3^1} + 1 = \frac{4}{3} \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $b_n + 1 = \frac{4}{3} \left(\frac{2}{3} \right)^{n-1} = 2 \left(\frac{2}{3} \right)^n$

$$b_n = 2 \left(\frac{2}{3} \right)^n - 1$$

$$\therefore a_n = 3^n b_n = 3^n \left[2 \left(\frac{2}{3} \right)^n - 1 \right] = 2^{n+1} - 3^n$$

Substituting $\frac{a_n}{3^n} = b_n$

and $\frac{a_{n+1}}{3^{n+1}} = b_{n+1}$ into

$$\frac{a_{n+1}}{3^{n+1}} = \frac{2}{3} \cdot \frac{a_n}{3^n} - \frac{1}{3}$$

Replacing both b_{n+1} and

$$b_n \text{ with } x, x = \frac{2}{3}x - \frac{1}{3}$$

$$\therefore x = -1$$

$(b_n + 1)$ is the geometric sequence with 1st term

$$b_1 + 1 = \frac{4}{3} \text{ and}$$

common ratio $\frac{2}{3}$.

(1) $a_1 = 1, a_{n+1} = 3a_n + 2^n$

[Sol] Dividing both sides of $a_{n+1} = 3a_n + 2^n$ by 2^{n+1} ,

$$\frac{a_{n+1}}{2^{n+1}} = \frac{3}{2} \cdot \frac{a_n}{2^n} + \frac{1}{2}$$

Let $b_n = \frac{a_n}{2^n}$. Since $b_{n+1} = \frac{3}{2}b_n + \frac{1}{2}$,

$$\begin{cases} b_{n+1} + 1 = \frac{3}{2}(b_n + 1) \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 + 1 = \frac{a_1}{2^1} + 1 = \frac{3}{2} \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $b_n + 1 = \frac{3}{2} \left(\frac{3}{2} \right)^{n-1} = \left(\frac{3}{2} \right)^n$

$$b_n = \left(\frac{3}{2} \right)^n - 1$$

$$\therefore a_n = 2^n b_n = 2^n \left[\left(\frac{3}{2} \right)^n - 1 \right] = 3^n - 2^n$$

N46b

$$(2) \quad a_1 = \frac{1}{8}, \quad a_{n+1} = 3^{n+1} - 5a_n$$

[Sol] Dividing both sides of $a_{n+1} = 3^{n+1} - 5a_n$ by 3^{n+1} ,

$$\frac{a_{n+1}}{3^{n+1}} = 1 - \frac{5}{3} \cdot \frac{a_n}{3^n}$$

Let $b_n = \frac{a_n}{3^n}$. Since $b_{n+1} = 1 - \frac{5}{3}b_n$,

$$\begin{cases} b_{n+1} - \frac{3}{8} = -\frac{5}{3} \left(b_n - \frac{3}{8} \right) \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 - \frac{3}{8} = \frac{a_1}{3^1} - \frac{3}{8} = -\frac{1}{3} \quad \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n - \frac{3}{8} = -\frac{1}{3} \left(-\frac{5}{3} \right)^{n-1}$

$$b_n = -\frac{1}{3} \left(-\frac{5}{3} \right)^{n-1} + \frac{3}{8}$$

$$\therefore a_n = 3^n b_n = 3^n \left[-\frac{1}{3} \left(-\frac{5}{3} \right)^{n-1} + \frac{3}{8} \right] = -(-5)^{n-1} + \frac{3^{n+1}}{8}$$

$$(3) \quad a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{2^n}$$

[Sol] Multiplying both sides of $a_{n+1} = a_n + \frac{1}{2^n}$ by 2^{n+1} ,

$$2^{n+1} a_{n+1} = 2 \cdot 2^n a_n + 2$$

Let $b_n = 2^n a_n$. Since $b_{n+1} = 2b_n + 2$,

$$\begin{cases} b_{n+1} + 2 = 2(b_n + 2) \quad \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 + 2 = 2^1 a_1 + 2 = 4 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n + 2 = 4 \cdot 2^{n-1} = 2^{n+1}$

$$b_n = 2^{n+1} - 2$$

$$\therefore a_n = \frac{b_n}{2^n} = \frac{1}{2^n} (2^{n+1} - 2) = 2 - \frac{1}{2^{n-1}}$$

Recurrence Relations

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Assume that the recurrence relation $a_{n+2} + pa_{n+1} + qa_n = 0 \dots ①$ is true where p and q are non-zero constants. By using two solutions α and β of quadratic equation $x^2 + px + q = 0$ where a_{n+2} , a_{n+1} and a_n are replaced with x^2 and 1 respectively, prove that the recurrence relation ① can be rearranged in $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$.

[Sol] Since two solutions of $x^2 + px + q = 0$ are α and β , from root-coefficient relationships,

$$\alpha + \beta = -p \dots ②, \quad \alpha\beta = q \dots ③$$

Substituting ② and ③ into ①,

$$a_{n+2} - (\alpha + \beta)a_{n+1} + \alpha\beta a_n = 0$$

$$a_{n+2} - \alpha a_{n+1} = \beta a_{n+1} - \alpha\beta a_n$$

$$\therefore a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n) \dots ④$$

From root-coefficient relationships, let two solutions of $ax^2 + bx + c = 0$ ($a \neq 0$) be α and β .

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a} \quad (J131)$$

④ is also true for the equation $a_{n+2} - \beta a_{n+1} = \alpha(a_{n+1} - \beta a_n)$ when α and β are switched.

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 0, a_2 = 2, a_{n+2} - 5a_{n+1} + 6a_n = 0$

[Sol] Rearranging $a_{n+2} - 5a_{n+1} + 6a_n = 0$,

$$\begin{cases} a_{n+2} - 2a_{n+1} = 3(a_{n+1} - 2a_n) \dots ① \\ a_{n+2} - 3a_{n+1} = 2(a_{n+1} - 3a_n) \dots ② \end{cases}$$

From ①, the sequence $\{a_{n+1} - 2a_n\}$ is the geometric sequence with 1st term $a_2 - 2a_1 = 2$ and common ratio 3.

$$\therefore a_{n+1} - 2a_n = 2 \cdot 3^{n-1} \dots ③$$

From ②, the sequence $\{a_{n+1} - 3a_n\}$ is the geometric sequence with 1st term $a_2 - 3a_1 = 2$ and common ratio 2.

$$\therefore a_{n+1} - 3a_n = 2 \cdot 2^{n-1} = 2^n \dots ④$$

From ③ and ④, $a_n = 2 \cdot 3^{n-1} - 2^n$

Replacing

a_{n+2}, a_{n+1}, a_n with $x^2, x, 1$ respectively
 $x^2 - 5x + 6 = 0$
 $(x-2)(x-3) = 0$
 $\therefore x = 2, 3$

$$(1) \quad a_1 = 1, a_2 = 5, a_{n+2} - 7a_{n+1} + 12a_n = 0$$

$$[\text{Sol}] \text{ Rearranging } a_{n+2} - 7a_{n+1} + 12a_n = 0,$$

$$\begin{cases} a_{n+2} - 3a_{n+1} = 4(a_{n+1} - 3a_n) \dots (1) \\ a_{n+2} - 4a_{n+1} = 3(a_{n+1} - 4a_n) \dots (2) \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n
with $x^2, x, 1$ respectively,
 $x^2 - 7x + 12 = 0$
 $(x-3)(x-4) = 0$
 $\therefore x = 3, 4$

From (1), the sequence $(a_{n+1} - 3a_n)$ is the geometric sequence with 1st term $a_2 - 3a_1 = 2$ and common ratio 4.

$$\therefore a_{n+1} - 3a_n = 2 \cdot 4^{n-1} \dots (3)$$

From (2), the sequence $(a_{n+1} - 4a_n)$ is the geometric sequence with 1st term $a_2 - 4a_1 = 1$ and common ratio 3.

$$\therefore a_{n+1} - 4a_n = 3^{n-1} \dots (4)$$

From (3) and (4), $a_n = 2 \cdot 4^{n-1} - 3^{n-1} \quad [= 2^{2n-1} - 3^{n-1}]$

$$(2) \quad a_1 = 0, a_2 = 1, 5a_{n+1} = 3a_{n+2} + 2a_n$$

$$[\text{Sol}] \text{ Since } 5a_{n+1} = 3a_{n+2} + 2a_n, \quad 3a_{n+2} - 5a_{n+1} + 2a_n = 0 \dots (1)$$

Rearranging (1),

$$\begin{cases} a_{n+2} - \frac{2}{3}a_{n+1} = a_{n+1} - \frac{2}{3}a_n \dots (2) \\ a_{n+2} - a_{n+1} = \frac{2}{3}(a_{n+1} - a_n) \dots (3) \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n
with $x^2, x, 1$ respectively,
 $3x^2 - 5x + 2 = 0$
 $(3x-2)(x-1) = 0$
 $\therefore x = \frac{2}{3}, 1$

From (2), the sequence $\left\{a_{n+1} - \frac{2}{3}a_n\right\}$ is the geometric sequence with 1st term

$$a_2 - \frac{2}{3}a_1 = 1 \text{ and common ratio } 1.$$

$$\therefore a_{n+1} - \frac{2}{3}a_n = 1 \dots (4)$$

From (3), the sequence $(a_{n+1} - a_n)$ is the geometric sequence with 1st term

$$a_2 - a_1 = 1 \text{ and common ratio } \frac{2}{3}.$$

$$\therefore a_{n+1} - a_n = \left(\frac{2}{3}\right)^{n-1} \dots (5)$$

From (4) and (5), $a_n = 3 \left[1 - \left(\frac{2}{3}\right)^{n-1} \right]$

Note Summary

The recurrence relation $a_{n+2} + pa_{n+1} + qa_n = 0$ can be rearranged into $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$ by using two solutions α and β of quadratic equation $x^2 + px + q = 0$. The sequence $(a_{n+1} - \alpha a_n)$ is a geometric sequence. Therefore, the general term of the sequence $\{a_n\}$ can be determined.

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Find the general terms of the following sequences $\{a_n\}$ and $\{b_n\}$ defined by following conditions.

Ex.

$$\begin{cases} a_1 = 2, b_1 = 1 \\ a_{n+1} = 3a_n + b_n \cdots \textcircled{1} \\ b_{n+1} = a_n + 3b_n \cdots \textcircled{2} \end{cases}$$

[Sol] From $\textcircled{1}$, $b_n = -3a_n + a_{n+1} \cdots \textcircled{3}$

From $\textcircled{3}$, $b_{n+1} = -3 \boxed{a_{n+1}} + \boxed{a_{n+2}} \cdots \textcircled{4}$ ← In $\textcircled{3}$, replacing n with $n+1$

Substituting $\textcircled{3}$ and $\textcircled{4}$ into $\textcircled{2}$,

$$\boxed{-3a_{n+1} + a_{n+2}} = a_n + 3(\boxed{-3a_n + a_{n+1}})$$

$$a_{n+2} - 6a_{n+1} + 8a_n = 0 \cdots \textcircled{5}$$

Rearranging $\textcircled{5}$,

$$\begin{cases} a_{n+2} - 2a_{n+1} = \boxed{4}(a_{n+1} - 2a_n) \cdots \textcircled{6} \\ a_{n+2} - 4a_{n+1} = \boxed{2}(a_{n+1} - 4a_n) \cdots \textcircled{7} \end{cases}$$

Replacing

$$\begin{aligned} & a_{n+2}, a_{n+1}, a_n \text{ with } \\ & x^2, x, 1 \text{ respective} \\ & x^2 - 6x + 8 = 0 \\ & (x-2)(x-4) = 0 \\ & \therefore x = 2, 4 \end{aligned}$$

From $\textcircled{6}$, the sequence $\{a_{n+1} - 2a_n\}$ is the geometric sequence with term $a_2 - 2a_1 = (3a_1 + b_1) - 2a_1 = 3$ and common ratio $\boxed{4}$.

$$\therefore a_{n+1} - 2a_n = 3 \cdot \boxed{4}^{n-1} \cdots \textcircled{8}$$

From $\textcircled{1}$,
 $a_2 = 3a_1$

From $\textcircled{7}$, the sequence $\{a_{n+1} - 4a_n\}$ is the geometric sequence with term $a_2 - 4a_1 = (3a_1 + b_1) - 4a_1 = -1$ and common ratio $\boxed{2}$.

$$\therefore a_{n+1} - 4a_n = -\boxed{2}^{n-1} \cdots \textcircled{9}$$

From $\textcircled{8}$ and $\textcircled{9}$, $a_n = \frac{1}{2}(3 \cdot \boxed{4}^{n-1} + \boxed{2}^{n-1}) \cdots \textcircled{10}$ ← From $\textcircled{8} - \textcircled{9}$,
 $2a_n = 3 \cdot 4^{n-1} - 2^{n-1}$

From $\textcircled{3}$ and $\textcircled{10}$, $b_n = -3 \cdot \frac{1}{2}(3 \cdot \boxed{4}^{n-1} + \boxed{2}^{n-1}) + \frac{1}{2}(3 \cdot \boxed{4}^n + \boxed{2}^n)$
 $= \frac{1}{2}(\boxed{3 \cdot 4^{n-1} - 2^{n-1}})$

Substituting $a_n = \frac{1}{2}(3 \cdot 4^{n-1} + 2^{n-1})$,
 $a_{n+1} = \frac{1}{2}(3 \cdot 4^n + 2^n)$ into
 $b_n = -3a_n + a_{n+1}$

$$\begin{cases} a_1 = 1, b_1 = 3 \\ a_{n+1} = 3a_n + b_n \quad \dots \textcircled{1} \\ b_{n+1} = 2a_n + 4b_n \quad \dots \textcircled{2} \end{cases}$$

sol] From $\textcircled{1}$, $b_n = -3a_n + a_{n+1} \quad \dots \textcircled{3}$

From $\textcircled{3}$, $b_{n+1} = -3a_{n+1} + a_{n+2} \quad \dots \textcircled{4}$

Substituting $\textcircled{3}$ and $\textcircled{4}$ into $\textcircled{2}$,

$$-3a_{n+1} + a_{n+2} = 2a_n + 4(-3a_n + a_{n+1})$$

$$a_{n+2} - 7a_{n+1} + 10a_n = 0 \quad \dots \textcircled{5}$$

Rearranging $\textcircled{5}$,

$$\begin{cases} a_{n+2} - 2a_{n+1} = 5(a_{n+1} - 2a_n) \quad \dots \textcircled{6} \\ a_{n+2} - 5a_{n+1} = 2(a_{n+1} - 5a_n) \quad \dots \textcircled{7} \end{cases}$$

From $\textcircled{6}$, the sequence $\{a_{n+1} - 2a_n\}$ is the geometric sequence with 1st term

$$a_2 - 2a_1 = (3a_1 + b_1) - 2a_1 = 4 \text{ and common ratio } 5.$$

$$\therefore a_{n+1} - 2a_n = 4 \cdot 5^{n-1} \quad \dots \textcircled{8}$$

From $\textcircled{7}$, the sequence $\{a_{n+1} - 5a_n\}$ is the geometric sequence with 1st term

$$a_2 - 5a_1 = (3a_1 + b_1) - 5a_1 = 1 \text{ and common ratio } 2.$$

$$\therefore a_{n+1} - 5a_n = 2^{n-1} \quad \dots \textcircled{9}$$

$$\text{From } \textcircled{8} \text{ and } \textcircled{9}, a_n = \frac{1}{3}(4 \cdot 5^{n-1} - 2^{n-1}) \quad \dots \textcircled{10}$$

$$\begin{aligned} \text{From } \textcircled{3} \text{ and } \textcircled{10}, b_n &= -3 \cdot \frac{1}{3}(4 \cdot 5^{n-1} - 2^{n-1}) + \frac{1}{3}(4 \cdot 5^n - 2^n) \\ &= \frac{1}{3}(8 \cdot 5^{n-1} + 2^{n-1}) \end{aligned}$$

Recurrence Relations

Name _____

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Time : to :

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1. The sequence $\{a_n\}$ is defined by the following conditions.

$$a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1} + 2a_n a_{n+1}}$$

(1) Let $b_n = \frac{1}{a_n}$. Express b_{n+2} in terms of b_{n+1} and b_n .

[Sol] Since $a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1} + 2a_n a_{n+1}}$,

$$\frac{1}{a_{n+2}} = \frac{2a_n - a_{n+1} + 2a_n a_{n+1}}{a_n a_{n+1}} = \frac{2}{a_{n+1}} - \frac{1}{a_n} + 2$$

$$\text{Let } b_n = \frac{1}{a_n}. \quad b_{n+2} = 2b_{n+1} - b_n + 2$$

(2) Let $b_{n+1} - b_n = c_n$. Express c_n in terms of n .

[Sol] Let $b_{n+1} - b_n = c_n$.

$$\begin{aligned} c_{n+1} &= b_{n+2} - b_{n+1} \\ &= (2b_{n+1} - b_n + 2) - b_{n+1} \\ &= b_{n+1} - b_n + 2 \\ &= c_n + 2 \quad \dots \textcircled{1} \end{aligned}$$

$$c_1 = b_2 - b_1 = \frac{1}{a_2} - \frac{1}{a_1} = 1 \quad \dots \textcircled{2}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, c_n = 1 + (n-1) \cdot 2 = 2n-1$$

From (1),

$$b_{n+2} = 2b_{n+1} - b_n + 2$$

$\{c_n\}$ is the arithmetic sequence with 1st term $c_1 = 1$ and common difference 2.

(3) Express a_n in terms of n .

[Sol] From (2), $b_{n+1} - b_n = 2n-1$

Since the general term of the sequence of differences of $\{b_n\}$ is $2n-1$ when $n \geq 2$,

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} (2k-1) = 2 + 2 \cdot \frac{1}{2} (n-1)n - (n-1) \\ &= n^2 - 2n + 3 \quad \dots \textcircled{3} \end{aligned}$$

Since $b_1 = 2$, $\textcircled{3}$ is also true when $n = 1$.

$$\therefore b_n = n^2 - 2n + 3$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{n^2 - 2n + 3}$$

149b

Given the sequence (a_n) , let S_n be the sum of the first n terms. Then,

$$S_1 = 1, S_2 = -13, S_{n+2} = -13S_{n+1} - 36S_n$$

are true. Express S_n and a_n in terms of n .

ol) Since $S_{n+2} = -13S_{n+1} - 36S_n$, $S_{n+2} + 13S_{n+1} + 36S_n = 0 \dots \textcircled{1}$

Rearranging $\textcircled{1}$,

$$\begin{cases} S_{n+2} + 9S_{n+1} = -4(S_{n+1} + 9S_n) \dots \textcircled{2} \\ S_{n+2} + 4S_{n+1} = -9(S_{n+1} + 4S_n) \dots \textcircled{3} \end{cases}$$

Replacing S_{n+2} , S_{n+1} , S_n
with x^2 , x , 1 respectively,
 $x^2 + 13x + 36 = 0$
 $(x+9)(x+4) = 0$
 $\therefore x = -9, -4$

From $\textcircled{2}$, the sequence $\{S_{n+1} + 9S_n\}$ is the geometric sequence with 1st term $S_2 + 9S_1 = -4$ and common ratio -4 .

$$\therefore S_{n+1} + 9S_n = -4(-4)^{n-1} = (-4)^n \dots \textcircled{4}$$

From $\textcircled{3}$, the sequence $\{S_{n+1} + 4S_n\}$ is the geometric sequence with 1st term $S_2 + 4S_1 = -9$ and common ratio -9 .

$$\therefore S_{n+1} + 4S_n = -9(-9)^{n-1} = (-9)^n \dots \textcircled{5}$$

From $\textcircled{4}$ and $\textcircled{5}$, $S_n = \frac{1}{5}[(-4)^n - (-9)^n]$

The 1st term a_1 is $a_1 = S_1 = 1$.

When $n \geq 2$,

$$a_n = S_n - S_{n-1}$$

Sum of a Sequence and
General Term (N34)

$$= \frac{1}{5}[(-4)^n - (-9)^n] - \frac{1}{5}[(-4)^{n-1} - (-9)^{n-1}]$$

$$= \frac{1}{5}[(-4-1)(-4)^{n-1} + (9+1)(-9)^{n-1}]$$

$$= -(-4)^{n-1} + 2(-9)^{n-1} \dots \textcircled{6}$$

Since $a_1 = 1$, $\textcircled{6}$ is also true when $n = 1$.

$$\therefore a_n = -(-4)^{n-1} + 2(-9)^{n-1}$$

Recurrence Relations

Name _____

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

(1) $a_1=6, a_{n+1}=a_n+4n$ ➡ N

[Sol] $a_{n+1}-a_n=4n$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n$,
when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} 4k = 6 + 4 \cdot \frac{1}{2}(n-1)n \\ &= 2(n^2 - n + 3) \quad \cdots \textcircled{1} \end{aligned}$$

Since $a_1=6$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = 2(n^2 - n + 3)$$

(2) $a_1=2, a_{n+1}=2a_n+5$ ➡ N

[Sol] Since $a_{n+1}=2a_n+5, a_{n+1}+5=2(a_n+5)$

Let $b_n = a_n + 5$.

$$\begin{cases} b_{n+1} = 2b_n & \cdots \textcircled{1} \\ b_1 = a_1 + 5 = 7 & \cdots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $b_n = 7 \cdot 2^{n-1}$

$$\therefore a_n = b_n - 5 = 7 \cdot 2^{n-1} - 5$$

N50b

$$(3) \quad a_1 = \frac{1}{3}, \quad a_{n+1} = \frac{a_n}{4a_n - 1}$$

[Sol] Since $a_{n+1} = \frac{a_n}{4a_n - 1}$, $\frac{1}{a_{n+1}} = \frac{4a_n - 1}{a_n} = 4 - \frac{1}{a_n}$

Let $b_n = \frac{1}{a_n}$. Since $b_{n+1} = -b_n + 4$,

$$\begin{cases} b_{n+1} - 2 = -(b_n - 2) \dots ① \\ b_1 - 2 = \frac{1}{a_1} - 2 = 1 \dots ② \end{cases}$$

From ① and ②, $b_n - 2 = (-1)^{n-1}$

$$b_n = (-1)^{n-1} + 2$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{(-1)^{n-1} + 2}$$

$$(4) \quad a_1 = 2, \quad a_2 = 1, \quad a_{n+2} = a_{n+1} + 6a_n$$

[Sol] Since $a_{n+2} = a_{n+1} + 6a_n$, $a_{n+2} - a_{n+1} - 6a_n = 0 \dots ①$

Rearranging ①,

$$\begin{cases} a_{n+2} + 2a_{n+1} = 3(a_{n+1} + 2a_n) \dots ② \\ a_{n+2} - 3a_{n+1} = -2(a_{n+1} - 3a_n) \dots ③ \end{cases}$$

From ②, the sequence $\{a_{n+1} + 2a_n\}$ is the geometric sequence with 1st term $a_2 + 2a_1 = 5$ and common ratio 3.

$$\therefore a_{n+1} + 2a_n = 5 \cdot 3^{n-1} \dots ④$$

From ③, the sequence $\{a_{n+1} - 3a_n\}$ is the geometric sequence with 1st term $a_2 - 3a_1 = -5$ and common ratio -2 .

$$\therefore a_{n+1} - 3a_n = -5(-2)^{n-1} \dots ⑤$$

From ④ and ⑤, $a_n = 3^{n-1} + (-2)^{n-1}$

Mathematical Induction

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Ex. Given that n is a natural number, prove the following equality.

$$2+4+6+\cdots+2n=n(n+1) \cdots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = \boxed{2}, \text{ RHS} = 1 \cdot (1+1) = \boxed{2}$$

Therefore, $\textcircled{1}$ is true when $n=1$.(ii) If $\textcircled{1}$ is true when $n=k$,

$$2+4+6+\cdots+2k = \boxed{k(k+1)} \cdots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & 2+4+6+\cdots+2k+2(k+1) \\ &= \boxed{k(k+1)} + 2(k+1) \end{aligned} \quad \leftarrow \text{From } \textcircled{2}$$

$$= (k+1)(\boxed{k+2})$$

$$= (k+1)[(\boxed{k+1})+1] \quad \leftarrow \text{The RHS of } \textcircled{1} \text{ when } n=k+1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .STEPS: 2. 2. $k(k+1)$, $k(k+1)$, $k+2$, $k+1$ The method of proof shown above is called **mathematical induction**.**Mathematical Induction**To prove that a proposition P is true for all natural numbers n by mathematical induction, the following two statements have to be proved.(i) P is true when $n=1$.(ii) If P is true when $n=k$, then P is also true when $n=k+1$.A statement that can be either true or false is called a **proposition**.

N51b

1. Given that n is a natural number, prove the following equality by mathematical induction.

$$1+3+5+\cdots+(2n-1)=n^2 \quad \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS}=1, \text{ RHS}=1^2=1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1+3+5+\cdots+(2k-1)=k^2 \quad \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$1+3+5+\cdots+(2k-1)+[2(k+1)-1]$$

$$=k^2+2k+1$$

$$=(k+1)^2$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Given that n is a natural number, prove the following equality by mathematical induction.

$$4 + 16 + 28 + \dots + (12n - 8) = 2n(3n - 1) \dots \textcircled{1}$$

sol] (i) When $n = 1$,

$$\text{LHS} = 4, \text{ RHS} = 2 \cdot 1 \cdot (3 \cdot 1 - 1) = 4$$

Therefore, $\textcircled{1}$ is true when $n = 1$.

(ii) If $\textcircled{1}$ is true when $n = k$,

$$4 + 16 + 28 + \dots + (12k - 8) = 2k(3k - 1) \dots \textcircled{2}$$

When $n = k + 1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$4 + 16 + 28 + \dots + (12k - 8) + [12(k + 1) - 8]$$

$$= 2k(3k - 1) + 12k + 4$$

$$= 6k^2 + 10k + 4$$

$$= 2(k + 1)(3k + 2)$$

$$= 2(k + 1)[3(k + 1) - 1]$$

Therefore, $\textcircled{1}$ is also true when $n = k + 1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathematical induction.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2) \quad \cdots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = 1 \cdot 2 = 2, \text{ RHS} = \frac{1}{3} \cdot 1 \cdot (1+1)(1+2) = 2$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2) \quad \cdots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)[(k+1)+1] \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)(k+3) \\ &= \frac{1}{3}(k+1)[(k+1)+1][(k+1)+2] \end{aligned}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

2b

Given that n is a natural number, prove the following equality by mathematical induction.

$$10 + 10^2 + 10^3 + \dots + 10^n = \frac{10}{9}(10^n - 1) \dots \textcircled{1}$$

1] (i) When $n=1$,

$$\text{LHS} = 10, \text{ RHS} = \frac{10}{9}(10^1 - 1) = 10$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$10 + 10^2 + 10^3 + \dots + 10^k = \frac{10}{9}(10^k - 1) \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$10 + 10^2 + 10^3 + \dots + 10^k + 10^{k+1}$$

$$= \frac{10}{9}(10^k - 1) + 10^{k+1}$$

$$= \frac{10}{9}(10^k - 1 + 9 \cdot 10^k)$$

$$= \frac{10}{9}(10^{k+1} - 1)$$

$$10^k + 9 \cdot 10^k = (1+9) \cdot 10^k = 10^{k+1}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathematical induction.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1} \quad \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = \frac{1}{1 \cdot 2} = \frac{1}{2}, \quad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)[(k+1)+1]} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1} \end{aligned}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

3b

Given that n is a natural number, prove the following equality by mathematical induction.

$$1 + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{n+1}{2^n} = 3 - \frac{n+3}{2^n} \dots \textcircled{1}$$

(i) When $n=1$,

$$\text{LHS} = 1, \text{ RHS} = 3 - \frac{1+3}{2^1} = 1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1 + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{k+1}{2^k} = 3 - \frac{k+3}{2^k} \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$1 + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{k+1}{2^k} + \frac{(k+1)+1}{2^{k+1}}$$

$$= 3 - \frac{k+3}{2^k} + \frac{k+2}{2^{k+1}}$$

$$= 3 - \frac{2(k+3) - (k+2)}{2^{k+1}}$$

$$= 3 - \frac{k+4}{2^{k+1}}$$

$$= 3 - \frac{(k+1)+3}{2^{k+1}}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Mathematical Induction

Name _____

Date _____

Time

100%	~90%	~80%	~70%	69%~
Competition 0	1	2	3	4

Ex. Given that n is a natural number, prove the following statement by mathematical induction.

$n^3 + 2n$ is a multiple of 3.

[Sol] Let the statement " $n^3 + 2n$ is a multiple of 3" be ①.

(i) When $n=1$,

$$n^3 + 2n = \boxed{1}^3 + 2 \cdot \boxed{1} = \boxed{3}$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$k^3 + 2k = 3m \dots (2) \quad \leftarrow k^3 + 2k \text{ is a multiple of 3.}$$

When $n = k + 1$, rearranging by using ②,

$$\begin{aligned} & (k+1)^3 + 2(k+1) \\ &= (k^3 + 3k^2 + 3k + 1) + (2k + 2) \\ &= (k^3 + 2k) + \boxed{3}(k^2 + k + 1) \\ &= \boxed{3}m + \boxed{3}(k^2 + k + 1) \\ &= \boxed{3}(m + k^2 + k + 1) \end{aligned}$$

Since $m + k^2 + k + 1$ is an integer,

$(k+1)^3 + 2(k+1)$ is a multiple of 3.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

N54b

1. Given that n is a natural number, prove the following statement by mathematical induction.

$2n^3 - 3n^2 + n + 6$ is a multiple of 6.

[Sol] Let the statement " $2n^3 - 3n^2 + n + 6$ is a multiple of 6" be ①.

(i) When $n=1$,

$$2n^3 - 3n^2 + n + 6 = 2 \cdot 1^3 - 3 \cdot 1^2 + 1 + 6 = 6$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$2k^3 - 3k^2 + k + 6 = 6m \quad \dots \textcircled{2}$$

When $n=k+1$, rearranging by using ②,

$$\begin{aligned} & 2(k+1)^3 - 3(k+1)^2 + (k+1) + 6 \\ &= 2k^3 + 6k^2 + 6k + 2 - 3k^2 - 6k - 3 + k + 1 + 6 \\ &= (2k^3 - 3k^2 + k + 6) + 6k^2 \\ &= 6m + 6k^2 \\ &= 6(m + k^2) \end{aligned}$$

Since $m + k^2$ is an integer,

$2(k+1)^3 - 3(k+1)^2 + (k+1) + 6$ is a multiple of 6.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

Mathematical Induction

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

1. Given that n is a natural number, prove the following statement by mathematical induction.

$$3^{2n} - 1 \text{ is a multiple of } 8.$$

[Sol] Let the statement " $3^{2n} - 1$ is a multiple of 8" be ①.

(i) When $n=1$,

$$3^{2n} - 1 = 3^2 - 1 = 8$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$3^{2k} - 1 = 8m \quad \dots \text{②}$$

When $n=k+1$, rearranging by using ②,

$$3^{2(k+1)} - 1$$

$$= 9 \cdot 3^{2k} - 1$$

$$= 9(8m+1) - 1$$

$$= 9 \cdot 8m + 8$$

$$= 8(9m+1)$$

Since $9m+1$ is an integer,

$3^{2(k+1)} - 1$ is a multiple of 8.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

$$3^{2(k+1)} = 3^{2k+2} = 3^2 \cdot 3^{2k}$$

From ②,
 $3^{2k} = 8m+1$

155b

Given that n is a natural number, prove the following statement by mathematical induction.

$11^{n+1} + 12^{2n-1}$ is a multiple of 19.

ol] Let the statement " $11^{n+1} + 12^{2n-1}$ is a multiple of 19" be ①.

(i) When $n=1$,

$$11^{n+1} + 12^{2n-1} = 11^2 + 12^1 = 133 = 19 \cdot 7$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$11^{k+1} + 12^{2k-1} = 19m \quad \dots \text{②}$$

When $n=k+1$, rearranging by using ②,

$$\begin{aligned} & 11^{(k+1)+1} + 12^{2(k+1)-1} \\ &= 11 \cdot 11^{k+1} + 12^{2k+1} \\ &= 11(19m - 12^{2k-1}) + 12^2 \cdot 12^{2k-1} \\ &= 11 \cdot 19m + 133 \cdot 12^{2k-1} \\ &= 19(11m + 7 \cdot 12^{2k-1}) \end{aligned}$$

Since $11m + 7 \cdot 12^{2k-1}$ is an integer,

$11^{(k+1)+1} + 12^{2(k+1)-1}$ is a multiple of 19.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

From ②,
 $11^{k+1} = 19m - 12^{2k-1}$

Mathematical Induction

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Ex. Given that n is a natural number greater than or equal to 3, prove the following inequality by mathematical induction.

$$3^n > 8n \quad \cdots \textcircled{1}$$

[Sol] (i) When $n=3$,

$$\text{LHS} = 3^3 = 27$$

$$\text{RHS} = 8 \cdot 3 = 24$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=3$.

(ii) For $k \geq 3$, if $\textcircled{1}$ is true when $n=k$,

$$3^k > 8k \quad \cdots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$3^{k+1} - 8(k+1) = 3 \cdot 3^k - 8(k+1)$$

$$> 3 \cdot 8k - 8(k+1)$$

$$= 8(2k-1) > 0$$

$$\therefore 3^{k+1} > 8(k+1)$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 3.

Proof of
LHS - RHS > 0
(J194)

From $\textcircled{2}$

Since $k \geq 3$,
 $2k-1 > 0$

GTS: 3, 24, $k+1$, $k+1$, $8k$, $k+1$, $2k-1$, $k+1$

As shown in **Ex.**, if the statement is proved when $n=3$, then from (i) and (ii), the statement is true for all natural numbers n greater than or equal to 3.

Given that n is a natural number greater than or equal to 3, prove the following inequality by mathematical induction.

$$2^n > 2n + 1 \quad \dots \textcircled{1}$$

Sol] (i) When $n=3$,

$$\text{LHS} = 2^3 = 8$$

$$\text{RHS} = 2 \cdot 3 + 1 = 7$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=3$.

(ii) For $k \geq 3$, if $\textcircled{1}$ is true when $n=k$,

$$2^k > 2k + 1 \quad \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$\begin{aligned} 2^{k+1} - [2(k+1) + 1] &= 2 \cdot 2^k - (2k+3) \\ &> 2(2k+1) - (2k+3) \\ &= 2k-1 > 0 \end{aligned}$$

$$\therefore 2^{k+1} > 2(k+1) + 1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 3.

Mathematical Induction

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex.

$$a_1 = 2, a_{n+1} = 2 - \frac{1}{a_n} \dots \textcircled{1}$$

[Sol] From the conditions,

$$a_1 = 2, a_2 = \frac{3}{2}, a_3 = \frac{4}{3}, a_4 = \frac{5}{4}, \dots$$

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2 - \frac{1}{a_1} \\ a_3 &= 2 - \frac{1}{a_2} \\ &\vdots \end{aligned}$$

Therefore, the general term of $\{a_n\}$ is assumed to be

$$a_n = \frac{n+1}{n} \dots \textcircled{2}$$

Prove that the assumption $\textcircled{2}$ is true by mathematical induction.

(i) When $n=1$, in $\textcircled{2}$,

$$a_1 = \frac{1+1}{1} = 2$$

Therefore, $\textcircled{2}$ is true when $n=1$.

It coincides with condition $a_1=2$.

(ii) If $\textcircled{2}$ is true when $n=k$,

$$a_k = \frac{k+1}{k}$$

When $n=k+1$, substituting $n=k$ into $\textcircled{1}$,

$$\begin{aligned} a_{k+1} &= 2 - \frac{1}{a_k} \\ &= 2 - \frac{k}{k+1} \\ &= \frac{k+2}{k+1} = \frac{(k+1)+1}{k+1} \end{aligned}$$

$$a_k = \frac{k+1}{k}$$

Therefore, $\textcircled{2}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{2}$ is true for all natural numbers n .

$$\therefore a_n = \frac{n+1}{n}$$

N57b

$$(1) \quad a_1 = 2, \quad a_{n+1} = \frac{a_n}{1+a_n} \quad \dots \textcircled{1}$$

[Sol] From the conditions,

$$a_1 = 2, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{2}{5}, \quad a_4 = \frac{2}{7}, \quad \dots$$

Therefore, the general term of (a_n) is assumed to be

$$a_n = \frac{2}{2n-1} \quad \dots \textcircled{2}$$

Prove that the assumption $\textcircled{2}$ is true by mathematical induction.

(i) When $n=1$, in $\textcircled{2}$,

$$a_1 = \frac{2}{2 \cdot 1 - 1} = 2$$

Therefore, $\textcircled{2}$ is true when $n=1$.

(ii) If $\textcircled{2}$ is true when $n=k$,

$$a_k = \frac{2}{2k-1}$$

When $n=k+1$, substituting $n=k$ into $\textcircled{1}$,

$$\begin{aligned} a_{k+1} &= \frac{a_k}{1+a_k} \\ &= \frac{\frac{2}{2k-1}}{1+\frac{2}{2k-1}} \\ &= \frac{2}{2k+1} = \frac{2}{2(k+1)-1} \end{aligned}$$

Therefore, $\textcircled{2}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{2}$ is true for all natural numbers n .

$$\therefore a_n = \frac{2}{2n-1}$$

Mathematical Induction

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Ex. Given that n is a natural number, compare the values of 2^n and $3n+1$.
(Fill in the gray boxes with either $>$ or $<$.)

[Sol] The table below shows the values of 2^n and $3n+1$ when $n=1, 2, 3, 4, 5$.

n	1	2	3	4	5
2^n	2	4	8	16	32
$3n+1$	4	7	10	13	16

From the table,

when $n \geq 4$, the following inequality is assumed to be true:

$$2^n > 3n+1 \quad \cdots \textcircled{1}$$

Prove that the assumption $\textcircled{1}$ is true by mathematical induction.

(i) $\textcircled{1}$ is true when $n=4$.

(ii) For $k \geq 4$, if $\textcircled{1}$ is true when $n=k$,

$$2^k > 3k+1 \quad \cdots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of then from $\textcircled{2}$,

$$\begin{aligned} 2^{k+1} - [3(k+1) + 1] &= 2 \cdot 2^k - (3k+4) \\ &> 2(3k+1) - (3k+4) \\ &= 3k-2 > 0 \end{aligned}$$

$$\therefore 2^{k+1} > 3(k+1) + 1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n when $n \geq 4$.

Thus, when $n=1, 2, 3$, $2^n < 3n+1$ and

when $n \geq 4$, $2^n > 3n+1$

1. Given that n is a natural number, compare the values of 3^n and $5n+1$.

[Sol] The table below shows the values of 3^n and $5n+1$ when $n=1, 2, 3, 4, 5$.

n	1	2	3	4	5
3^n	3	9	27	81	243
$5n+1$	6	11	16	21	26

From the table,

when $n \geq 3$, the following inequality is assumed to be true:

$$3^n > 5n+1 \quad \dots \textcircled{1}$$

Prove that the assumption $\textcircled{1}$ is true by mathematical induction.

(i) $\textcircled{1}$ is true when $n=3$.

(ii) For $k \geq 3$, if $\textcircled{1}$ is true when $n=k$,

$$3^k > 5k+1 \quad \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$\begin{aligned} 3^{k+1} - [5(k+1)+1] &= 3 \cdot 3^k - (5k+6) \\ &> 3(5k+1) - (5k+6) \\ &= 10k-3 > 0 \end{aligned}$$

$$\therefore 3^{k+1} > 5(k+1)+1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n when $n \geq 3$.

Thus, when $n=1, 2$, $3^n < 5n+1$ and

when $n \geq 3$, $3^n > 5n+1$

Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathematical induction.

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \dots + (n-1) \cdot 2 + n \cdot 1 = \frac{1}{6}n(n+1)(n+2)$$

[Sol] Let $1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \dots + (n-1) \cdot 2 + n \cdot 1 = \frac{1}{6}n(n+1)(n+2)$

(i) When $n=1$,

$$\text{LHS} = 1 \cdot 1 = 1, \text{ RHS} = \frac{1}{6} \cdot 1 \cdot (1+1)(1+2) = 1$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$1 \cdot k + 2 \cdot (k-1) + 3 \cdot (k-2) + \dots + (k-1) \cdot 2 + k \cdot 1 = \frac{1}{6}k(k+1)(k+2)$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$1 \cdot (k+1) + 2 \cdot [(k+1)-1] + 3 \cdot [(k+1)-2] + \dots + [(k+1)-1] \cdot 2 + (k+1) \cdot 1$$

$$= [1 \cdot k + 2 \cdot (k-1) + 3 \cdot (k-2) + \dots + k \cdot 1] + 1 + 2 + 3 + \dots + k + (k+1)$$

$$= \frac{1}{6}k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2)$$

$$= \frac{1}{6}(k+1)(k+2)(k+3)$$

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}(k+1)[(k+1)+1][(k+1)+2]$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

※	$1 \cdot (k+1) + 2 \cdot k$	$+ 3 \cdot (k-1) + \dots + k \cdot 2 + (k+1) \cdot 1$
-)	$1 \cdot k$	$+ 2 \cdot (k-1) + 3 \cdot (k-2) + \dots + k \cdot 1$
	1	+ 2 + 3 + ... + k + (k+1)

Given that n is a natural number greater than or equal to 2, prove the following inequality by mathematical induction.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \frac{2n}{n+1} \quad \dots \textcircled{1}$$

(i) When $n=2$,

$$\text{LHS} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

$$\text{RHS} = \frac{2 \cdot 2}{2+1} = \frac{4}{3}$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=2$.

(ii) For $k \geq 2$, if $\textcircled{1}$ is true when $n=k$,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} > \frac{2k}{k+1} \quad \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$\begin{aligned} & \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1} \right) - \frac{2(k+1)}{(k+1)+1} \\ & > \left(\frac{2k}{k+1} + \frac{1}{k+1} \right) - \frac{2(k+1)}{k+2} \quad \text{From } \textcircled{2} \\ & = \frac{2k+1}{k+1} - \frac{2(k+1)}{k+2} \\ & = \frac{(2k+1)(k+2) - 2(k+1)^2}{(k+1)(k+2)} \\ & = \frac{k}{(k+1)(k+2)} > 0 \quad \leftarrow \text{Since } k \geq 2 \end{aligned}$$

$$\therefore \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1} > \frac{2(k+1)}{(k+1)+1}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 2.

Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathematical induction. ➡ N!

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1) \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = 1^2 = 1, \text{ RHS} = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2 \cdot 1 + 1) = 1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{1}{6}k(k+1)(2k+1) \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1] \end{aligned}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

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2. Given that n is a natural number greater than or equal to 2, prove the following inequality by mathematical induction.

$$3^n > 4n \dots \textcircled{1}$$

[Sol] (i) When $n=2$,

$$\text{LHS} = 3^2 = 9$$

$$\text{RHS} = 4 \cdot 2 = 8$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=2$.

- (ii) For $k \geq 2$, if $\textcircled{1}$ is true when $n=k$,

$$3^k > 4k \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$3^{k+1} - 4(k+1) = 3 \cdot 3^k - 4(k+1)$$

$$> 3 \cdot 4k - 4(k+1)$$

$$= 4(2k-1) > 0$$

$$\therefore 3^{k+1} > 4(k+1)$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 2.

Infinite Sequences

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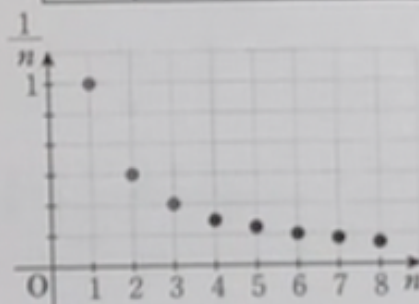
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A sequence with infinite terms $a_1, a_2, a_3, \dots, a_n, \dots$ is called an *infinite sequence*, and it is expressed as $\{a_n\}$.

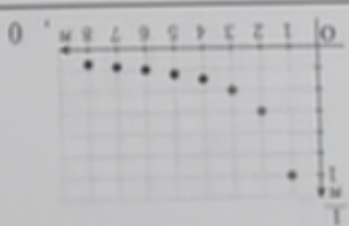
Given the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$, fill in the following table and plot the points representing the values of $\frac{1}{n}$ when $n=4, 5, 6, 7, 8$ on the coordinate plane below. Then, consider how the value of $\frac{1}{n}$ changes as n increases infinitely.

[Sol] When $n=1, 2, 3, 4, 5, 10, 100$, the values of $\frac{1}{n}$ are as shown in the table below.

n	1	2	3	4	5	...	10	...	100	...
$\frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...	$\frac{1}{10}$	...	$\frac{1}{100}$	...



Therefore, the value of $\frac{1}{n}$ decreases and approaches 0 as n increases infinitely.



...	$\frac{1}{100}$	...	$\frac{1}{10}$	...	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{n}$
...	100	...	10	...	5	4	3	2	1	n

Given the sequence $\{a_n\}$, if a_n approaches a constant value α as n approaches infinity, then $\{a_n\}$ is said to *converge* to α , where it is expressed as follows:

$$\lim_{n \rightarrow \infty} a_n = \alpha \quad \text{or} \quad a_n \rightarrow \alpha \text{ as } n \rightarrow \infty$$

The value α is called the *limit value* of $\{a_n\}$. In other words, the *limit* of $\{a_n\}$ is α . If the value of all terms of the sequence is constant c , then the limit value is also considered as c and expressed as follows: $\lim_{n \rightarrow \infty} c = c$

The symbol ∞ is read as "infinity" and represents an unbounded quantity that is greater than every real number.

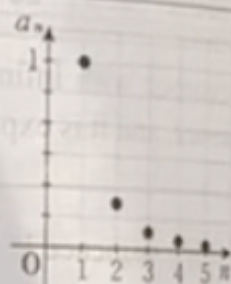
N61b

For each given sequence $\{a_n\}$, plot the points representing the values of a_n when $n=1, 2, 3, 4, 5$ on the coordinate plane, and then find the limit value.

Ex.

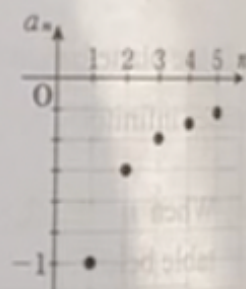
$$1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$$

$$[\text{Sol}] \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$



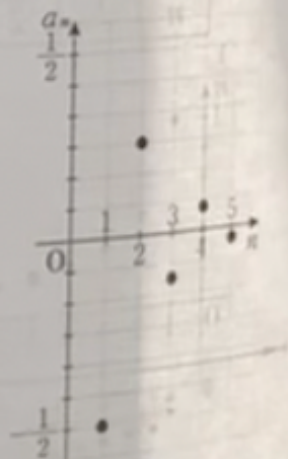
$$(1) -1, -\frac{1}{2}, -\frac{1}{3}, \dots, -\frac{1}{n}, \dots$$

$$[\text{Sol}] \lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$$



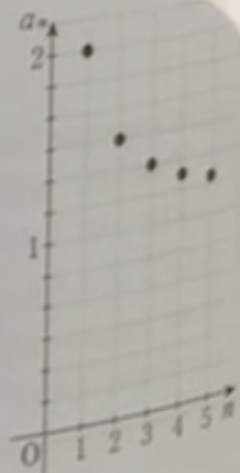
$$(2) -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots, \left(-\frac{1}{2}\right)^n, \dots$$

$$[\text{Sol}] \lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^n = 0$$



$$(3) 2, \frac{3}{2}, \frac{4}{3}, \dots, 1 + \frac{1}{n}, \dots$$

$$[\text{Sol}] \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$$



Infinite Sequences

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When the sequence $\{a_n\}$ does not converge, $\{a_n\}$ is said to **diverge**.

When $\{a_n\}$ **diverges to positive infinity**, it is said that the limit of $\{a_n\}$ is positive infinity and is expressed as follows: $\lim_{n \rightarrow \infty} a_n = \infty$ or $a_n \rightarrow \infty$ as $n \rightarrow \infty$

When $\{a_n\}$ **diverges to negative infinity**, it is said that the limit of $\{a_n\}$ is negative infinity and is expressed as follows: $\lim_{n \rightarrow \infty} a_n = -\infty$ or $a_n \rightarrow -\infty$ as $n \rightarrow \infty$

When a divergent sequence does not diverge to positive or negative infinity, the sequence is said to **oscillate**.

1. For each given sequence $\{a_n\}$, plot the points representing the values of a_n with $n=4, 5$ on the coordinate plane, and then find the limit. (When it oscillates, write "Oscillates.")

(1) $1, 2, 4, \dots, 2^{n-1}, \dots$

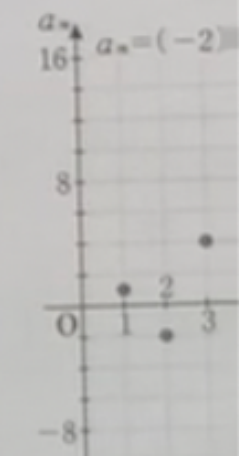
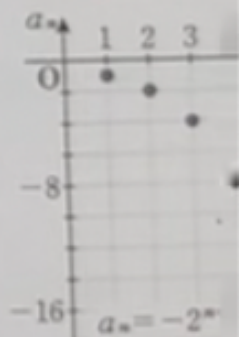
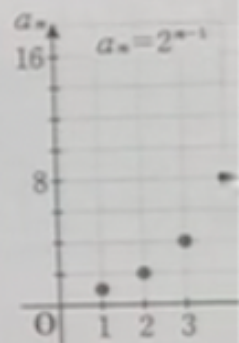
[Sol] $\lim_{n \rightarrow \infty} 2^{n-1} = \infty$

(2) $-1, -2, -4, \dots, -2^{n-1}, \dots$

[Sol] $\lim_{n \rightarrow \infty} (-2^{n-1}) = -\infty$

(3) $1, -2, 4, \dots, (-2)^{n-1}, \dots$

[Sol] Oscillates



Limit of a Sequence

$\left\{ \begin{array}{l} \text{Converges} \\ \\ \text{Diverges} \end{array} \right.$	$\lim_{n \rightarrow \infty} a_n = \alpha$	(converges to a constant value α) ...①
	$\lim_{n \rightarrow \infty} a_n = \infty$	(diverges to positive infinity) ...②
	$\lim_{n \rightarrow \infty} a_n = -\infty$	(diverges to negative infinity) ...③
	Oscillates	(no limit) ...④

2. Classify each given limit by indicating the appropriate number ① to ④.
If the limit is classified as ①, state the limit value.

Ex. $\lim_{n \rightarrow \infty} \frac{2}{n}$ (①, limit value 0) $\lim_{n \rightarrow \infty} (-3)^n$ (④)

(1) $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1}$ (①, limit value 0) (6) $\lim_{n \rightarrow \infty} (-5)^n$ (④)

(2) $\lim_{n \rightarrow \infty} \left(2 + \frac{1}{n} \right)$ (①, limit value 2) (7) $\lim_{n \rightarrow \infty} -4^n$ (③)

(3) $\lim_{n \rightarrow \infty} (2n + 5)$ (②) (8) $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ (①, limit value 0)

(4) $\lim_{n \rightarrow \infty} (-2n + 5)$ (③) (9) $\lim_{n \rightarrow \infty} \left(n - \frac{1}{n} \right)$ (②)

(5) $\lim_{n \rightarrow \infty} \cos \frac{\pi}{n}$ (①, limit value 1) (10) $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{2}$ (④)

Substituting $n = 10, 100, \dots$

$$\cos \frac{\pi}{10}, \cos \frac{\pi}{100}, \dots$$

$\lim_{n \rightarrow \infty} \cos \frac{\pi}{n}$ approaches $\cos 0$.

Substituting $n = 1, 2, 3, 4, 5, \dots$

$$\sin \frac{\pi}{2}, \sin \pi, \sin \frac{3\pi}{2}, \sin 2\pi, \sin \frac{5\pi}{2}, \dots$$

$\therefore 1, 0, -1, 0, 1, \dots$

Infinite Sequences

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Properties of Limits of Sequences

When the sequences $\{a_n\}$ and $\{b_n\}$ converge, where $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$.

$$\lim_{n \rightarrow \infty} k a_n = k \alpha$$

(k is a constant)

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \alpha + \beta,$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \alpha - \beta$$

$$\lim_{n \rightarrow \infty} a_n b_n = \alpha \beta$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}$$

 $(\beta \neq 0)$

Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{2n+1}{3n^2+n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{1}{n^2}}{3 + \frac{1}{n}} = 0$$

Dividing the numerator and the denominator by the term with the highest power in the denominator (n^2 in this case)

$$(1) \quad \lim_{n \rightarrow \infty} \frac{2n+3}{4n^2+5n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{3}{n^2}}{4 + \frac{5}{n}} = 0$$

$$(2) \quad \lim_{n \rightarrow \infty} \frac{8-5n}{4+n} = \lim_{n \rightarrow \infty} \frac{\frac{8}{n} - 5}{\frac{4}{n} + 1} = -5$$

$$(3) \quad \lim_{n \rightarrow \infty} \frac{n^2+5n+4}{3-2n^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n} + \frac{4}{n^2}}{\frac{3}{n^2} - 2} = -\frac{1}{2}$$

N63b

$$4) \lim_{n \rightarrow \infty} \frac{2n^2 - 3n}{n+1} = \lim_{n \rightarrow \infty} \frac{2n-3}{1+\frac{1}{n}} = \infty$$

$$5) \lim_{n \rightarrow \infty} \frac{-2n^3 - 5n}{n^2 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{-2n - \frac{5}{n}}{1 + \frac{2}{n} + \frac{1}{n^2}} = -\infty$$

$$6) \lim_{n \rightarrow \infty} \frac{-2n^2(n+2)}{2n^3-5} = \lim_{n \rightarrow \infty} \frac{-2n^3-4n^2}{2n^3-5} = \lim_{n \rightarrow \infty} \frac{-2-\frac{4}{n}}{2-\frac{5}{n^3}} = -1$$

$$7) \lim_{n \rightarrow \infty} \frac{5+n+n^5}{n(4-3n^2)} = \lim_{n \rightarrow \infty} \frac{5+n+n^5}{4n-3n^3} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n^3} + \frac{1}{n^2} + n^2}{\frac{4}{n^2} - 3} = -\infty$$

$$8) \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+1}}{2n} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n^2}}}{2} = \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left(\frac{5}{n^3} + \frac{1}{n^2} + n^2 \right) = \infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{4}{n^2} - 3 \right) = -3$$

$$\frac{\sqrt{n^2+1}}{n} = \sqrt{\frac{n^2+1}{n^2}} = \sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}} = \sqrt{1 + \frac{1}{n^2}}$$

$$9) \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2+3n}}{6n} = \lim_{n \rightarrow \infty} \frac{\sqrt{4+\frac{3}{n}}}{6} = \frac{1}{3}$$

$$10) \lim_{n \rightarrow \infty} \frac{2n^2+4n+3}{\sqrt{3n^2+1}} = \lim_{n \rightarrow \infty} \frac{2n+4+\frac{3}{n}}{\sqrt{3+\frac{1}{n^2}}} = \infty$$

Ex. since $\lim_{n \rightarrow \infty} (2n+1) = \infty$ and $\lim_{n \rightarrow \infty} (2n+1) = \infty$, the limit becomes an indeterminate form $\frac{\infty}{\infty}$. Therefore, the expression has to be rearranged.

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Find the following limits.

Ex.

$$\begin{aligned}\lim_{n \rightarrow \infty} \log_2 \frac{16n+1}{2n} &= \lim_{n \rightarrow \infty} \log_2 \frac{16 + \frac{1}{n}}{2} \\ &= \log_2 8 \\ &= 3\end{aligned}$$

$$\log_a a^m = m$$

$$\begin{aligned}(1) \quad \lim_{n \rightarrow \infty} \log_3 \frac{9n+1}{3n} &= \lim_{n \rightarrow \infty} \log_3 \frac{9 + \frac{1}{n}}{3} \\ &= \log_3 3 \\ &= 1\end{aligned}$$

$$\log_a a = 1$$

$$\begin{aligned}(2) \quad \lim_{n \rightarrow \infty} [\log_3(2n^2-1) - \log_3(6n^2+2)] &= \lim_{n \rightarrow \infty} \log_3 \frac{2n^2-1}{6n^2+2} \leftarrow \log_a M - \log_a N = \log_a \frac{M}{N} \\ &= \lim_{n \rightarrow \infty} \log_3 \frac{2 - \frac{1}{n^2}}{6 + \frac{2}{n^2}} \\ &= \log_3 \frac{1}{3} \\ &= -1\end{aligned}$$

$$\begin{aligned}(3) \quad \lim_{n \rightarrow \infty} [\log_2 \sqrt{4n^4+3n} - \log_2(2n^2+n)] &= \lim_{n \rightarrow \infty} \log_2 \frac{\sqrt{4n^4+3n}}{2n^2+n} \\ &= \lim_{n \rightarrow \infty} \log_2 \frac{\sqrt{4 + \frac{3}{n^3}}}{2 + \frac{1}{n}} \\ &= \log_2 1 \\ &= 0\end{aligned}$$

$$\log_a 1 = 0$$

$$\log_a a = 1, \quad \log_a 1 = 0, \quad \log_a \frac{M}{N} = \log_a M - \log_a N$$

N64b

Ex.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{3n^2} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{2}n(n+1)}{3n^2} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{6n} \\ &= \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{6} \\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}(4) \quad \lim_{n \rightarrow \infty} \frac{1^2+2^2+3^2+\dots+n^2}{n^3} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2+3n+1}{6n^2} \\ &= \lim_{n \rightarrow \infty} \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6} \\ &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}(5) \quad \lim_{n \rightarrow \infty} \frac{2^3+4^3+6^3+\dots+(2n)^3}{n^4} &= \lim_{n \rightarrow \infty} \frac{8\left[\frac{1}{2}n(n+1)\right]^2}{n^4} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2+4n+2}{n^2} \\ &= \lim_{n \rightarrow \infty} \left(2+\frac{4}{n}+\frac{2}{n^2}\right) \\ &= 2\end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

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Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} (n^2 - 3n) = \lim_{n \rightarrow \infty} n^2 \left(1 - \frac{3}{n} \right) = \infty$$

← Taking out the term with the highest power

$$(1) \quad \lim_{n \rightarrow \infty} (n^2 - 10n) = \lim_{n \rightarrow \infty} n^2 \left(1 - \frac{10}{n} \right) = \infty$$

$$(2) \quad \lim_{n \rightarrow \infty} (2n^3 - 3n^2 + 4) = \lim_{n \rightarrow \infty} n^3 \left(2 - \frac{3}{n} + \frac{4}{n^3} \right) = \infty$$

$$(3) \quad \lim_{n \rightarrow \infty} (4n^2 - 2n^3) = \lim_{n \rightarrow \infty} n^3 \left(\frac{4}{n} - 2 \right) = -\infty$$

←

$$\begin{aligned} \lim_{n \rightarrow \infty} n^3 &= \infty \\ \lim_{n \rightarrow \infty} \left(\frac{4}{n} - 2 \right) &= -2 \end{aligned}$$

$$(4) \quad \lim_{n \rightarrow \infty} (n - \sqrt{n}) = \lim_{n \rightarrow \infty} n \left(1 - \frac{1}{\sqrt{n}} \right) = \infty$$

$$(5) \quad \lim_{n \rightarrow \infty} (\sqrt{n^3} - n^2) = \lim_{n \rightarrow \infty} n^2 \left(\frac{1}{\sqrt{n}} - 1 \right) = -\infty$$

In **Ex.**, since $\lim_{n \rightarrow \infty} n^2 = \infty$ and $\lim_{n \rightarrow \infty} 3n = \infty$, the limit becomes an indeterminate form $\infty - \infty$. Therefore, in such cases, the expression has to be rearranged.

N65b

Ex.

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + 3n} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + 3n} - n)(\sqrt{n^2 + 3n} + n)}{\sqrt{n^2 + 3n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{n^2 + 3n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{\sqrt{1 + \frac{3}{n}} + 1}$$

$$= \frac{3}{2}$$

Considering

$\sqrt{n^2 + 3n} - n = \frac{\sqrt{n^2 + 3n} - n}{1}$
then multiplying the numerator and the denominator by $\sqrt{n^2 + 3n} + n$

$$(6) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n + 1} - n)(\sqrt{n^2 + n + 1} + n)}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + 1}$$

$$= \frac{1}{2}$$

$$(7) \lim_{n \rightarrow \infty} (\sqrt{2n} + \sqrt{n})(\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} \frac{(\sqrt{2n} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{2n} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{2n} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{2} + 1}{\sqrt{1 + \frac{1}{n}} + 1}$$

$$= \frac{\sqrt{2} + 1}{2}$$

Dividing the numerator and the denominator by $\sqrt{n}$

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Find the following limits.

Ex.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2+5n}-n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+5n}+n}{(\sqrt{n^2+5n}-n)(\sqrt{n^2+5n}+n)} \\
 &= \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+5n}+n}{5n} \\
 &= \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{5}{n}}+1}{5} \\
 &= \frac{2}{5}
 \end{aligned}$$

Multiplying the numerator and the denominator by $\sqrt{n^2+5n}+n$

$$\begin{aligned}
 (1) \quad \lim_{n \rightarrow \infty} \frac{3}{\sqrt{4n^2+2n}-2n} &= \lim_{n \rightarrow \infty} \frac{3(\sqrt{4n^2+2n}+2n)}{(\sqrt{4n^2+2n}-2n)(\sqrt{4n^2+2n}+2n)} \\
 &= \lim_{n \rightarrow \infty} \frac{3(\sqrt{4n^2+2n}+2n)}{2n} \\
 &= \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{4+\frac{2}{n}}+2\right)}{2} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \lim_{n \rightarrow \infty} \frac{5}{\sqrt{n^2+2}-\sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{5(\sqrt{n^2+2}+\sqrt{n})}{(\sqrt{n^2+2}-\sqrt{n})(\sqrt{n^2+2}+\sqrt{n})} \\
 &= \lim_{n \rightarrow \infty} \frac{5(\sqrt{n^2+2}+\sqrt{n})}{n^2-n+2} \\
 &= \lim_{n \rightarrow \infty} \frac{5\left(\sqrt{\frac{1}{n^2}+\frac{2}{n^4}}+\sqrt{\frac{1}{n^3}}\right)}{1-\frac{1}{n}+\frac{2}{n^2}} \\
 &= 0
 \end{aligned}$$

N66b

$$(3) \lim_{n \rightarrow \infty} \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{1+\frac{1}{n}} + 1\right)}{\sqrt{1+\frac{3}{n}} + 1}$$

$$= 3$$

Multiplying the numerator and the denominator by $\sqrt{n+1} + \sqrt{n}$

Considering $\sqrt{n+3} - \sqrt{n}$, then multiplying the numerator and the denominator by $\sqrt{n+3} + \sqrt{n}$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})(\sqrt{n+3} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{1+\frac{1}{n}} + 1\right)}{\sqrt{1+\frac{3}{n}} + 1}$$

$$= 3$$

$$\lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+2} - \sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}}$$

$$\lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}}$$

$$\lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1+\frac{2}{n}} + 1\right)}{\sqrt{1+\frac{5}{n}} + \sqrt{1+\frac{3}{n}}}$$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})(\sqrt{n+5} + \sqrt{n+3})}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}} = \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1+\frac{2}{n}} + 1\right)}{\sqrt{1+\frac{5}{n}} + \sqrt{1+\frac{3}{n}}}$$

$$= 2$$

N66b

$$(3) \lim_{n \rightarrow \infty} \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{1+\frac{1}{n}} + 1\right)}{\sqrt{1+\frac{3}{n}} + 1}$$

$$= 3$$

Multiplying the numerator and the denominator by $\sqrt{n+1} + \sqrt{n}$

Considering

$\sqrt{n+3} - \sqrt{n} = \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$
then multiplying the numerator and the denominator by $\sqrt{n+3} + \sqrt{n}$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})(\sqrt{n+3} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{1+\frac{1}{n}} + 1\right)}{\sqrt{1+\frac{3}{n}} + 1}$$

$$= 3$$

$$(4) \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}}$$

$$= \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1+\frac{2}{n}} + 1\right)}{\sqrt{1+\frac{5}{n}} + \sqrt{1+\frac{3}{n}}}$$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})(\sqrt{n+5} + \sqrt{n+3})}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}} = \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1+\frac{2}{n}} + 1\right)}{\sqrt{1+\frac{5}{n}} + \sqrt{1+\frac{3}{n}}}$$

$$= 2$$

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Limits of Sequences and Their Relationships

1. For all n , when $a_n \leq b_n$,if $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$, then $\alpha \leq \beta$ if $\lim_{n \rightarrow \infty} a_n = \infty$, then $\lim_{n \rightarrow \infty} b_n = \infty$ 2. For all n , when $a_n \leq c_n \leq b_n$,if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \alpha$, then $\lim_{n \rightarrow \infty} c_n = \alpha$

Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$$

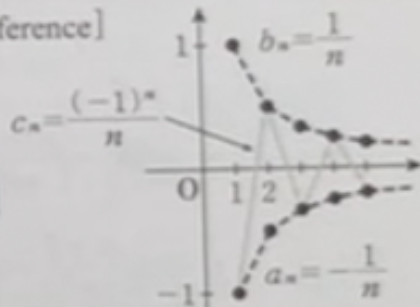
[Reference]

[Sol] Since $-1 \leq (-1)^n \leq 1$,

$$-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

 $n > 0$ Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$



$$(1) \lim_{n \rightarrow \infty} \frac{(-1)^n}{\sqrt{n}}$$

[Sol] Since $-1 \leq (-1)^n \leq 1$,

$$-\frac{1}{\sqrt{n}} \leq \frac{(-1)^n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{\sqrt{n}}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\sqrt{n}} = 0$$

As for Limits of Sequences and Their Relationships, statement 1 is also true when $a_n < b_n$
 statement 2 is also true when $a_n \leq c_n < b_n$, $a_n < c_n \leq b_n$, $a_n < c_n < b_n$.

N67b

$$(2) \lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

[Sol] Since $-1 \leq \sin \frac{n\pi}{2} \leq 1$,

$$-\frac{1}{n} \leq \frac{1}{n} \sin \frac{n\pi}{2} \leq \frac{1}{n}$$



$$-1 \leq \sin \theta \leq 1 \quad (\text{M131})$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{n\pi}{2} = 0$$

$$(3) \lim_{n \rightarrow \infty} \frac{n+1}{n^2} \cos n\theta \quad (\theta \text{ is a constant})$$

[Sol] Since $-1 \leq \cos n\theta \leq 1$,

$$-\frac{n+1}{n^2} \leq \frac{n+1}{n^2} \cos n\theta \leq \frac{n+1}{n^2}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{n+1}{n^2}\right) = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} - \frac{1}{n^2}\right) = 0$ and

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = 0,$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} \cos n\theta = 0$$

$$(4) \lim_{n \rightarrow \infty} \frac{2^n}{n} \quad \left(\text{Using inequality } 2^n > \frac{n^2}{2}\right)$$

[Sol] Since $2^n > \frac{n^2}{2}$,

$$\frac{2^n}{n} > \boxed{\frac{n}{2}}$$

Then, since $\lim_{n \rightarrow \infty} \frac{n}{2} = \infty$,

$$\lim_{n \rightarrow \infty} \frac{2^n}{n} = \infty$$

Infinite Sequences

Narrie

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Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} (n + \sin n\theta)$$

[Sol] $\lim_{n \rightarrow \infty} \frac{1}{2n-1} (n + \sin n\theta)$

$$= \lim_{n \rightarrow \infty} \left(\frac{n}{2n-1} + \frac{1}{2n-1} \sin n\theta \right)$$

Then, $\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \boxed{\frac{1}{2}} \dots \textcircled{1} \quad \leftarrow \lim_{n \rightarrow \infty} \frac{n}{2n-1}$
converges.

Also, since $-1 \leq \sin n\theta \leq 1$,

$$-\frac{1}{2n-1} \leq \frac{1}{2n-1} \sin n\theta \leq \frac{1}{2n-1} \quad \leftarrow \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \begin{matrix} 2n-1 > 0 \\ 2n-1 > 0 \\ 2n-1 > 0 \end{matrix}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{2n-1} \right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} \sin n\theta = \boxed{0} \quad \dots \textcircled{2} \quad \leftarrow \dots \lim_{n \rightarrow \infty} \frac{1}{2n-1} \sin n\theta \text{ converges.}$$

From ① and ②,

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} (n + \sin n\theta) = \boxed{\frac{1}{2}} \leftarrow \lim_{n \rightarrow \infty} \left(\frac{n}{2n-1} + \frac{1}{2n-1} \sin n\theta \right)$$

From ① and ②,

$$\lim_{n \rightarrow \infty} \left(\frac{n}{2n-1} + \frac{1}{2n-1} \sin n\theta \right) \\ = \lim_{n \rightarrow \infty} \frac{n}{2n-1} + \lim_{n \rightarrow \infty} \frac{1}{2n-1} \sin$$

N68b

$$(1) \lim_{n \rightarrow \infty} \frac{1}{5n-2} \left(n + \cos \frac{n\pi}{6} \right)$$

$$\begin{aligned} \text{[Sol]} \quad & \lim_{n \rightarrow \infty} \frac{1}{5n-2} \left(n + \cos \frac{n\pi}{6} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{5n-2} + \frac{1}{5n-2} \cos \frac{n\pi}{6} \right) \end{aligned}$$

$$\text{Then, } \lim_{n \rightarrow \infty} \frac{n}{5n-2} = \lim_{n \rightarrow \infty} \frac{1}{5 - \frac{2}{n}} = \frac{1}{5} \dots \textcircled{1}$$

$$\text{Also, since } -1 \leq \cos \frac{n\pi}{6} \leq 1,$$

$$-\frac{1}{5n-2} \leq \frac{1}{5n-2} \cos \frac{n\pi}{6} \leq \frac{1}{5n-2}$$

$$\text{Then, since } \lim_{n \rightarrow \infty} \left(-\frac{1}{5n-2} \right) = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{1}{5n-2} = 0,$$

$$\lim_{n \rightarrow \infty} \frac{1}{5n-2} \cos \frac{n\pi}{6} = 0 \dots \textcircled{2}$$

From ① and ②,

$$\lim_{n \rightarrow \infty} \frac{1}{5n-2} \left(n + \cos \frac{n\pi}{6} \right) = \frac{1}{5} + 0 = \frac{1}{5}$$

Infinite Sequences

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1. Given the sequence $\{a_n\}$: $a_1=5$, $a_2=12$, $a_3=21$, $a_4=32$, ..., when the sequence of differences $\{b_n\}$ of this sequence becomes an arithmetic sequence, solve the following questions.

- (1) Find a_n .

[Sol] $\{b_n\}$ is 7, 9, 11,

$$\therefore b_n = 7 + (n-1) \cdot 2 = 2n + 5$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (2k+5) \\ &= 5 + 2 \cdot \frac{1}{2} (n-1)n + 5(n-1) \\ &= n^2 + 4n \quad \cdots \textcircled{1} \end{aligned}$$

Since $a_1=5$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = n^2 + 4n$$

$$\{a_n\}: 5 \quad 12 \quad 21 \quad 32 \dots$$

$$\{b_n\}: 7 \quad 9 \quad 11 \dots$$

$\{b_n\}$ is the arithmetic sequence with 1st term 7 and common difference 2.

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

(N32)

- (2) Find $\lim_{n \rightarrow \infty} (\sqrt{a_n} - n)$ and $\lim_{n \rightarrow \infty} \left(\sqrt{\frac{a_n}{n}} - \sqrt{n} \right)$.

[Sol] From (1),

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{a_n} - n) &= \lim_{n \rightarrow \infty} (\sqrt{n^2 + 4n} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + 4n} - n)(\sqrt{n^2 + 4n} + n)}{\sqrt{n^2 + 4n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{4n}{\sqrt{n^2 + 4n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{4}{n}} + 1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sqrt{\frac{a_n}{n}} - \sqrt{n} \right) &= \lim_{n \rightarrow \infty} \left(\sqrt{\frac{n^2 + 4n}{n}} - \sqrt{n} \right) \\ &= \lim_{n \rightarrow \infty} (\sqrt{n+4} - \sqrt{n}) \\ &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+4} - \sqrt{n})(\sqrt{n+4} + \sqrt{n})}{\sqrt{n+4} + \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{4}{\sqrt{n+4} + \sqrt{n}} \\ &= 0 \end{aligned}$$

N69b

2. Given that n is a positive integer and $x \geq 0$, solve the following.
- (1) Using inequality $(1+x)^n \geq 1+nx+\frac{1}{2}n(n-1)x^2$, prove that $1+\sqrt{\frac{2}{n}} > n^{\frac{1}{n}}$ is true.

Sol] In $(1+x)^n \geq 1+nx+\frac{1}{2}n(n-1)x^2$, let $x = \sqrt{\frac{2}{n}}$.

$$\left(1+\sqrt{\frac{2}{n}}\right)^n \geq 1+n\sqrt{\frac{2}{n}}+\frac{1}{2}n(n-1)\cdot\frac{2}{n}$$

$$= 1+\sqrt{2n}+n-1$$

$$= n+\sqrt{2n} > n \quad \leftarrow \sqrt{2n} > 0$$

$$\therefore \left(1+\sqrt{\frac{2}{n}}\right)^n > n$$

$1+\sqrt{\frac{2}{n}} > 0$, $n > 0$; therefore,

$$1+\sqrt{\frac{2}{n}} > n^{\frac{1}{n}}$$

When $a > 0$, $b > 0$,
 $a^b < b^b \iff a < b$
 (K188)

n is a positive integer and $x \geq 0$, solve the following questions.

Equality $(1+x)^n \geq 1 + nx + \frac{1}{2}n(n-1)x^2$, prove that

$\frac{2}{n} > n^{\frac{1}{n}}$ is true.

$$(1+x)^n \geq 1 + nx + \frac{1}{2}n(n-1)x^2, \text{ let } x = \sqrt{\frac{2}{n}}.$$

$$\left(1 + \sqrt{\frac{2}{n}}\right)^n \geq 1 + n\sqrt{\frac{2}{n}} + \frac{1}{2}n(n-1) \cdot \frac{2}{n}$$

$$= 1 + \sqrt{2n} + n - 1$$

$$= n + \sqrt{2n} > n \quad \leftarrow \sqrt{2n} > 0$$

$$\left(1 + \sqrt{\frac{2}{n}}\right)^n > n$$

> 0 , $n > 0$; therefore,

$$\frac{2}{n} > n^{\frac{1}{n}}$$

When $a > 0$, $b > 0$ and $p > 0$,
 $a^p < b^p \Leftrightarrow a < b$
 (K188)

Find the value of $\lim_{n \rightarrow \infty} n^{\frac{1}{n}}$.

sol] Since $n \geq 1$, $n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1 \dots \textcircled{1}$

← For all positive integers n ,
 $1^{\frac{1}{n}} = 1$

From $\textcircled{1}$ and (1),

$$1 \leq n^{\frac{1}{n}} < 1 + \sqrt{\frac{2}{n}}$$

Then, since $\lim_{n \rightarrow \infty} 1 = 1$ and $\lim_{n \rightarrow \infty} \left(1 + \sqrt{\frac{2}{n}}\right) = 1$, ← $\lim_{n \rightarrow \infty} c = c$ (c is a constant)

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Infinite Sequences

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Find the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{5-2n^2}{3n^2+4} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n^2}-2}{3+\frac{4}{n^2}} = -\frac{2}{3}$$

⇒ N

$$(2) \lim_{n \rightarrow \infty} \frac{2n^2+3n+1}{12n^3-18n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{3}{n^2} + \frac{1}{n^3}}{12 - \frac{18}{n^2}} = 0$$

⇒ N

$$(3) \lim_{n \rightarrow \infty} \frac{n^2+6}{\sqrt{5n^2+4}} = \lim_{n \rightarrow \infty} \frac{n + \frac{6}{n}}{\sqrt{5 + \frac{4}{n^2}}} = \infty$$

⇒ N

$$(4) \lim_{n \rightarrow \infty} (n^3 - 6n^2) = \lim_{n \rightarrow \infty} n^3 \left(1 - \frac{6}{n}\right) = \infty$$

⇒ N

$$(5) \lim_{n \rightarrow \infty} (\sqrt{n^2 - 9n} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - 9n} - n)(\sqrt{n^2 - 9n} + n)}{\sqrt{n^2 - 9n} + n} \Rightarrow \text{N65}$$

$$= \lim_{n \rightarrow \infty} \frac{-9n}{\sqrt{n^2 - 9n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-9}{\sqrt{1 - \frac{9}{n}} + 1}$$

$$= -\frac{9}{2}$$

$$(6) \lim_{n \rightarrow \infty} \frac{4}{\sqrt{n^2 + 3n} - n} = \lim_{n \rightarrow \infty} \frac{4(\sqrt{n^2 + 3n} + n)}{(\sqrt{n^2 + 3n} - n)(\sqrt{n^2 + 3n} + n)} \Rightarrow \text{N66}$$

$$= \lim_{n \rightarrow \infty} \frac{4(\sqrt{n^2 + 3n} + n)}{3n}$$

$$= \lim_{n \rightarrow \infty} \frac{4\left(\sqrt{1 + \frac{3}{n}} + 1\right)}{3}$$

$$= \frac{8}{3}$$

$\Rightarrow \text{N67}$

$$(7) \lim_{n \rightarrow \infty} \frac{1}{n} \cos \frac{n\pi}{3}$$

[Sol] Since $-1 \leq \cos \frac{n\pi}{3} \leq 1$,

$$-\frac{1}{n} \leq \frac{1}{n} \cos \frac{n\pi}{3} \leq \frac{1}{n}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \cos \frac{n\pi}{3} = 0$$

Infinite Geometric Sequences

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A sequence $a, ar, ar^2, \dots, ar^{n-1}, \dots$ is called an *infinite geometric sequence* with 1st term a and common ratio r .

Find the limit for each given infinite geometric sequence. (When it oscillates write "Oscillates.")

[1] $2, 4, 8, 16, \dots, 2^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} 2^n = \boxed{\infty}$

[2] $1, 1, 1, 1, \dots, 1^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} 1^n = \boxed{1}$

[3] $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \left(\frac{1}{2}\right)^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = \boxed{0}$

[4] $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots, \left(-\frac{1}{3}\right)^n, \dots$

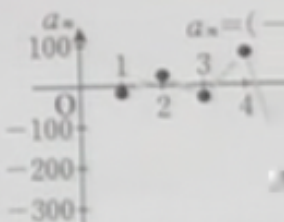
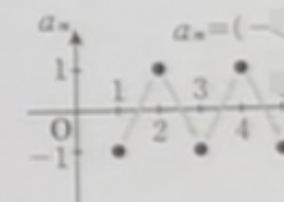
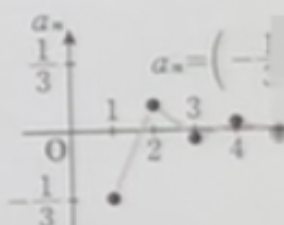
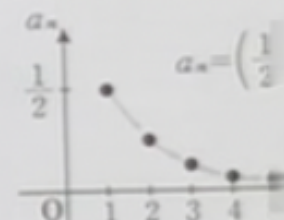
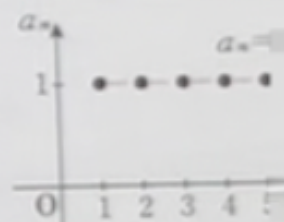
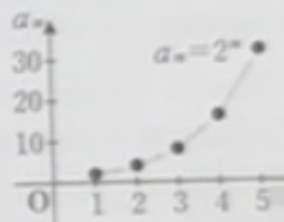
[Sol] $\lim_{n \rightarrow \infty} \left(-\frac{1}{3}\right)^n = \boxed{0}$

[5] $-1, 1, -1, 1, \dots, (-1)^n, \dots$

[Sol] **Oscillates**

[6] $-3, 9, -27, 81, \dots, (-3)^n, \dots$

[Sol] **Oscillates**



N71b

Limit of an Infinite Geometric Sequence $\{r^n\}$

When $r > 1$,	$\lim_{n \rightarrow \infty} r^n = \infty$	Diverges
When $r = 1$,	$\lim_{n \rightarrow \infty} r^n = 1$	Diverges
When $ r < 1$,	$\lim_{n \rightarrow \infty} r^n = 0$	Converges
When $r \leq -1$,	Oscillates (no limit)	Diverges

Classify each given limit by indicating the appropriate number ① to ④

- Converges ...①
- Diverges to positive infinity ...②
- Diverges to negative infinity ...③
- Oscillates (no limit) ...④

If the limit is classified as ①, state the limit value.

Ex. $\lim_{n \rightarrow \infty} \left(-\frac{1}{5}\right)^n$ $\leftarrow \left|-\frac{1}{5}\right| < 1$ (①, limit value 0)

(1) $\lim_{n \rightarrow \infty} \left(-\frac{2}{3}\right)^n$ (①, limit value 0)

(5) $\lim_{n \rightarrow \infty} \left(\frac{\sqrt{5}}{2}\right)^n$
 Since $\sqrt{5} > 2$, $\frac{\sqrt{5}}{2} > 1$

(2) $\lim_{n \rightarrow \infty} \left(\frac{5}{4}\right)^n$ (②)

(6) $\lim_{n \rightarrow \infty} \left[1 + \left(\frac{\sqrt{3}}{2}\right)^n\right]$ (①, limit value 1)

(3) $\lim_{n \rightarrow \infty} \left[-\left(\frac{5}{3}\right)^n\right]$ (③)

(7) $\lim_{n \rightarrow \infty} \left[-\left(\frac{3}{2}\right)^{n+1}\right]$
 $-\left(\frac{3}{2}\right)^{n+1} = -\frac{3}{2} \left(\frac{3}{2}\right)^n$

$\lim_{n \rightarrow \infty} \left(-\frac{4}{3}\right)^n$ (④)

(8) $\lim_{n \rightarrow \infty} \left[\left(\frac{1}{2}\right)^n + (-\sqrt{2})^n\right]$
 $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$ and $\lim_{n \rightarrow \infty} (-\sqrt{2})^n$ oscillates.

Infinite Geometric Sequences

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Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{1+3^n}{3^n+4^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{4}\right)^n + \left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1} = 0 \quad \leftarrow$$

Dividing the numerator and the denominator by the term with the largest absolute value of the base in the denominator (4^n in this case)

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1+2^n}{2^n+3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n}{\left(\frac{2}{3}\right)^n + 1} = 0$$

$$(2) \quad \lim_{n \rightarrow \infty} \frac{(-2)^n + 2 \cdot 3^n}{3^n + 1} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{2}{3}\right)^n + 2}{1 + \left(\frac{1}{3}\right)^n} = 2$$

$$(3) \quad \lim_{n \rightarrow \infty} \frac{3^{n+1}}{4^n + 2^n} = \lim_{n \rightarrow \infty} \frac{3\left(\frac{3}{4}\right)^n}{1 + \left(\frac{1}{2}\right)^n} = 0$$

$$(4) \quad \lim_{n \rightarrow \infty} \frac{5^n - 3^{n-1}}{5^n + 3^n} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{3}\left(\frac{3}{5}\right)^n}{1 + \left(\frac{3}{5}\right)^n} = 1$$

$$(5) \quad \lim_{n \rightarrow \infty} \frac{3^{n-1} + 4^{n+1}}{3^n - 4^n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{3}\left(\frac{3}{4}\right)^n + 4}{\left(\frac{3}{4}\right)^n - 1} = -4$$

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$$(6) \lim_{n \rightarrow \infty} \frac{4^n}{2^n + 3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{4}{3}\right)^n}{\left(\frac{2}{3}\right)^n + 1} = \infty$$

$$(7) \lim_{n \rightarrow \infty} \frac{-5^n + 3^{n+1}}{3^n + 4^n} = \lim_{n \rightarrow \infty} \frac{-\left(\frac{5}{4}\right)^n + 3\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1} = -\infty$$

$$(8) \lim_{n \rightarrow \infty} \frac{1 + (-3)^n}{2^n - (-3)^n} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{1}{3}\right)^n + 1}{\left(-\frac{2}{3}\right)^n - 1} = -1$$

$$(9) \lim_{n \rightarrow \infty} \frac{4^n + (-8)^n}{(-6)^n + 5^n} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{2}{3}\right)^n + \left(\frac{4}{3}\right)^n}{1 + \left(-\frac{5}{6}\right)^n} = \infty$$

$$\lim_{n \rightarrow \infty} \frac{9^{n+1} + 3^n}{3^{2n} - (-5)^n} = \lim_{n \rightarrow \infty} \frac{9^{n+1} + 3^n}{9^n - (-5)^n} = \lim_{n \rightarrow \infty} \frac{9 + \left(\frac{1}{3}\right)^n}{1 - \left(-\frac{5}{9}\right)^n} = 9$$

$$\lim_{n \rightarrow \infty} \frac{7 + 4^{2n}}{6^n - 2} = \lim_{n \rightarrow \infty} \frac{7 + 16^n}{6^n - 2} = \lim_{n \rightarrow \infty} \frac{7\left(\frac{1}{6}\right)^n + \left(\frac{8}{3}\right)^n}{1 - 2\left(\frac{1}{6}\right)^n} = \infty$$

Infinite Geometric Sequences

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Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} (3^n - 2^n) = \lim_{n \rightarrow \infty} 3^n \left[1 - \left(\frac{2}{3} \right)^n \right] = \infty \quad \leftarrow$$

Taking out the term with the largest absolute value of the base

$$(1) \quad \lim_{n \rightarrow \infty} (4^n - 3^n) = \lim_{n \rightarrow \infty} 4^n \left[1 - \left(\frac{3}{4} \right)^n \right] = \infty$$

$$(2) \quad \lim_{n \rightarrow \infty} (3^n - 5^n) = \lim_{n \rightarrow \infty} 5^n \left[\left(\frac{3}{5} \right)^n - 1 \right] = -\infty$$

$$(3) \quad \lim_{n \rightarrow \infty} [4^n + (-2)^n] = \lim_{n \rightarrow \infty} 4^n \left[1 + \left(-\frac{1}{2} \right)^n \right] = \infty$$

$$(4) \quad \lim_{n \rightarrow \infty} (2^{3n} - 3^{2n}) = \lim_{n \rightarrow \infty} (8^n - 9^n) = \lim_{n \rightarrow \infty} 9^n \left[\left(\frac{8}{9} \right)^n - 1 \right] = -\infty$$

$$(5) \quad \lim_{n \rightarrow \infty} \frac{5^n - 6^n}{2^{2n}} = \lim_{n \rightarrow \infty} \frac{6^n \left[\left(\frac{5}{6} \right)^n - 1 \right]}{4^n} = \lim_{n \rightarrow \infty} \left(\frac{3}{2} \right)^n \left[\left(\frac{5}{6} \right)^n - 1 \right] = -\infty$$

[Alternative Solution $\lim_{n \rightarrow \infty} \frac{5^n - 6^n}{2^{2n}} = \lim_{n \rightarrow \infty} \left[\left(\frac{5}{4} \right)^n - \left(\frac{3}{2} \right)^n \right] = \lim_{n \rightarrow \infty} \left(\frac{3}{2} \right)^n \left[\left(\frac{5}{6} \right)^n - 1 \right] = -\infty$]

$$(6) \quad \lim_{n \rightarrow \infty} \frac{4^n - 3^{n+1}}{2^n} = \lim_{n \rightarrow \infty} \frac{4^n \left[1 - 3 \left(\frac{3}{4} \right)^n \right]}{2^n} = \lim_{n \rightarrow \infty} 2^n \left[1 - 3 \left(\frac{3}{4} \right)^n \right] = \infty$$

[Alternative Solution $\lim_{n \rightarrow \infty} \frac{4^n - 3^{n+1}}{2^n} = \lim_{n \rightarrow \infty} \left[2^n - 3 \left(\frac{3}{2} \right)^n \right] = \lim_{n \rightarrow \infty} 2^n \left[1 - 3 \left(\frac{3}{4} \right)^n \right] = \infty$]

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$$\lim_{n \rightarrow \infty} \frac{2 \cdot 2^2 \cdot \dots \cdot 2^n}{3^{n(n+1)/2}} = \lim_{n \rightarrow \infty} \frac{2^{1+2+\dots+n}}{3^{n(n+1)/2}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}n(n+1)}}{9^{\frac{1}{2}n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{9} \right)^{\frac{1}{2}n(n+1)}$$

$$= 0$$

Rearranging into the form $\frac{A^n}{B^n}$ so that the value $\frac{A}{B}$ is the same

$$\begin{aligned} 2^{1+2+\dots+n} &= 2^{\frac{1}{2}n(n+1)} \\ 3^{n(n+1)/2} &= 9^{\frac{1}{2}n(n+1)} \end{aligned}$$

$$7) \lim_{n \rightarrow \infty} \frac{2 \cdot 2^2 \cdot \dots \cdot 2^n}{8^{n(n+1)/2}} = \lim_{n \rightarrow \infty} \frac{2^{1+2+\dots+n}}{8^{n(n+1)/2}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}n(n+1)}}{64^{\frac{1}{2}n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{32} \right)^{\frac{1}{2}n(n+1)}$$

$$= 0$$

$$8) \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16 \cdot 16^2 \cdot \dots \cdot 16^n} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16^{1+2+\dots+n}}$$

$$= \lim_{n \rightarrow \infty} \frac{16^{\frac{1}{2}n(n+1)}}{16^{\frac{1}{2}n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} 1$$

$$= 1$$

$$\begin{aligned} 2^{2n(n+1)} &= 4^{n(n+1)} \\ &= 4^{\frac{1}{2}n(n+1)} \\ &= 16^{\frac{1}{2}n(n+1)} \end{aligned}$$

$\lim_{n \rightarrow \infty} c = c$ (c is a constant)

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16 \cdot 16^2 \cdot \dots \cdot 16^n} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16^{1+2+\dots+n}} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{2^{2n(n+1)}} = \lim_{n \rightarrow \infty} 1 = 1$$

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Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} \quad (r \neq -1)$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} = \frac{0}{1+0} = 0$$

$$\leftarrow \lim_{n \rightarrow \infty} r^n = 0$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} = \frac{1}{1+1} = \frac{1}{2}$$

$$\leftarrow \lim_{n \rightarrow \infty} r^n = 1$$

(iii) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{r}\right)^n + 1} = \frac{1}{0+1} = 1$$

$$\leftarrow \lim_{n \rightarrow \infty} \left(\frac{1}{r}\right)^n = 0$$

$$(1) \lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} \quad (r \neq -1)$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} = \frac{1-0}{1+0} = 1$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} = \frac{1-1}{1+1} = 0$$

(iii) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{r}\right)^n - 1}{\left(\frac{1}{r}\right)^n + 1} = \frac{0-1}{0+1} = -1$$

In **Ex.** above, when $r = -1$, the denominator becomes 0 if n is an odd number. Therefore the condition $r \neq -1$ is added.

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$$(2) \quad \lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} \quad (r \neq -1)$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} = \frac{0 - 1}{0 + 1} = -1$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} = \frac{1 - 1}{1 + 1} = 0$$

(iii) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{1}{r}\right)^n}{1 + \left(\frac{1}{r}\right)^n} = \frac{r - 0}{1 + 0} = r$$

$$(3) \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1}$$

(Solving for four different cases: $|r| < 1$, $r=1$, $r=-1$ and $|r| > 1$)

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{0 - 1}{0 + 1} = -1$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{1 - 1}{1 + 1} = 0$$

(iii) When $r = -1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{-1 - 1}{1 + 1} = -1$$

When m is a natural number
 $(-1)^{2m} = 1$
 $(-1)^{2m+1} = -1$

(iv) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{1}{r}\right)^{2n}}{1 + \left(\frac{1}{r}\right)^{2n}} = \frac{r - 0}{1 + 0} = r$$

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Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} \quad (r \neq -4)$$

[Sol] (i) When $|r| < 4$,

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{r}{4}\right)^n + \left(\frac{1}{2}\right)^n}{r\left(\frac{r}{4}\right)^n + 1} = \frac{0+0}{r \cdot 0 + 1} = 0$$

(ii) When $r = 4$,

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{4^n + 2^n}{4^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{4^n \left[1 + \left(\frac{1}{2}\right)^n\right]}{5 \cdot 4^n} = \frac{1+0}{5} = \frac{1}{5}$$

(iii) When $|r| > 4$,

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{1 + \left(\frac{2}{r}\right)^n}{r + \left(\frac{4}{r}\right)^n} = \frac{1+0}{r+0} = \frac{1}{r}$$

$$(1) \lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} \quad (r \neq -6)$$

[Sol] (i) When $|r| < 6$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} = \lim_{n \rightarrow \infty} \frac{r\left(\frac{r}{6}\right)^n + \left(\frac{1}{2}\right)^n}{\left(\frac{r}{6}\right)^n + 1} = \frac{r \cdot 0 + 0}{0 + 1} = 0$$

(ii) When $r = 6$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} = \lim_{n \rightarrow \infty} \frac{6^{n+1} + 3^n}{6^n + 6^n} = \lim_{n \rightarrow \infty} \frac{6^n \left[6 + \left(\frac{1}{2}\right)^n\right]}{2 \cdot 6^n} = \frac{6+0}{2} = \frac{3}{1}$$

(iii) When $|r| > 6$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} = \lim_{n \rightarrow \infty} \frac{r + \left(\frac{3}{r}\right)^n}{1 + \left(\frac{6}{r}\right)^n} = \frac{r+0}{1+0} = r$$

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$$(2) \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}}$$

[Sol] (i) When $|r| < 5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{r \left(\frac{r}{5} \right)^{2n} - 1}{\left(\frac{r}{5} \right)^{2n} + 1} = \frac{r \cdot 0 - 1}{0 + 1} = -1$$

(ii) When $r = 5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{5^{2n+1} - 5^{2n}}{5^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{4 \cdot 5^{2n}}{2 \cdot 5^{2n}} = 2$$

(iii) When $r = -5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{(-5)^{2n+1} - 5^{2n}}{(-5)^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{-6 \cdot 5^{2n}}{2 \cdot 5^{2n}} = -3 \leftarrow$$

$$(-5)^{2n} = 5^{2n}$$

(iv) When $|r| > 5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{5}{r} \right)^{2n}}{1 + \left(\frac{5}{r} \right)^{2n}} = \frac{r - 0}{1 + 0} = r$$

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Ex. Find the range of values of x for which the sequence $((x-1)^n)$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x-1$, $-1 < x-1 \leq 1$ ←

$$\therefore 0 < x \leq 2$$

Also, the limit values are,

when $0 < x < 2$, $\lim_{n \rightarrow \infty} (x-1)^n = 0$ and ← When $|r| < 1$, $\lim_{n \rightarrow \infty} r^n = 0$

when $x = 2$, $\lim_{n \rightarrow \infty} (x-1)^n = 1$ ← When $r = 1$, $\lim_{n \rightarrow \infty} r^n = 1$

The sequence (r^n) converges when $-1 < r \leq 1$.

1. Find the range of values of x for which the sequence $((x+2)^n)$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x+2$, $-1 < x+2 \leq 1$

$$\therefore -3 < x \leq -1$$

Also, the limit values are,

when $-3 < x < -1$, $\lim_{n \rightarrow \infty} (x+2)^n = 0$ and

when $x = -1$, $\lim_{n \rightarrow \infty} (x+2)^n = 1$

2. Find the range of values of x for which the sequence $\left\{\left(\frac{x+1}{3}\right)^n\right\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $\frac{x+1}{3}$, $-1 < \frac{x+1}{3} \leq 1$

$$\therefore -4 < x \leq 2$$

Also, the limit values are,

when $-4 < x < 2$, $\lim_{n \rightarrow \infty} \left(\frac{x+1}{3}\right)^n = 0$ and

when $x = 2$, $\lim_{n \rightarrow \infty} \left(\frac{x+1}{3}\right)^n = 1$

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3. Find the range of values of x for which the sequence $((x^2 - x - 1)^n)$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x^2 - x - 1$, $-1 < x^2 - x - 1 < 1$
 So, $-1 < x^2 - x - 1 \dots \textcircled{1}$ and also $x^2 - x - 1 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 - x > 0$; therefore, $x(x-1) > 0$

$$\therefore x < 0, 1 < x \dots \textcircled{3}$$

From $\textcircled{2}$, $x^2 - x - 2 < 0$; therefore, $(x+1)(x-2) < 0$

$$\therefore -1 < x < 2 \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$-1 < x < 0, 1 < x < 2$$

Also, the limit values are,

when $-1 < x < 0, 1 < x < 2$, $\lim_{n \rightarrow \infty} (x^2 - x - 1)^n = 0$ and

when $x = -1, 2$, $\lim_{n \rightarrow \infty} (x^2 - x - 1)^n = 1$

4. Find the range of values of x for which the sequence $((x^2 - 3x + 1)^n)$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x^2 - 3x + 1$, $-1 < x^2 - 3x + 1 < 1$

So, $-1 < x^2 - 3x + 1 \dots \textcircled{1}$ and also $x^2 - 3x + 1 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 - 3x + 2 > 0$; therefore, $(x-1)(x-2) > 0$

$$\therefore x < 1, 2 < x \dots \textcircled{3}$$

From $\textcircled{2}$, $x^2 - 3x < 0$; therefore, $x(x-3) < 0$

$$\therefore 0 < x < 3 \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$0 < x < 1, 2 < x < 3$$

Also, the limit values are,

when $0 < x < 1, 2 < x < 3$, $\lim_{n \rightarrow \infty} (x^2 - 3x + 1)^n = 0$ and

when $x = 0, 3$, $\lim_{n \rightarrow \infty} (x^2 - 3x + 1)^n = 1$

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1. Find the range of values of x for which the sequence $\left\{ \left(\frac{x^2+2x-5}{x^2-x+2} \right)^n \right\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $\frac{x^2+2x-5}{x^2-x+2}$, $-1 < \frac{x^2+2x-5}{x^2-x+2} \leq 1$

Since $x^2-x+2 > 0$, rearranging the inequality, ←

$$-(x^2-x+2) < x^2+2x-5 \leq x^2-x+2$$

$$\begin{aligned} x^2-x+2 \\ = \left(x-\frac{1}{2}\right)^2 + \frac{7}{4} > 0 \end{aligned}$$

So,

$$-(x^2-x+2) < x^2+2x-5 \cdots \textcircled{1} \text{ and also } x^2+2x-5 \leq x^2-x+2 \cdots \textcircled{2}$$

From $\textcircled{1}$, $2x^2+x-3 > 0$; therefore, $(2x+3)(x-1) > 0$

$$\therefore x < -\frac{3}{2}, 1 < x \cdots \textcircled{3}$$

From $\textcircled{2}$, $3x \leq 7$

$$\therefore x \leq \frac{7}{3} \cdots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$x < -\frac{3}{2}, 1 < x \leq \frac{7}{3}$$

Also, the limit values are,

$$\text{when } x < -\frac{3}{2}, 1 < x < \frac{7}{3}, \lim_{n \rightarrow \infty} \left(\frac{x^2+2x-5}{x^2-x+2} \right)^n = 0 \text{ and}$$

$$\text{when } x = \frac{7}{3}, \lim_{n \rightarrow \infty} \left(\frac{x^2+2x-5}{x^2-x+2} \right)^n = 1$$

Given $a_n = (4\sin^2\theta + 2\cos\theta - 3)^n$, find the range of values of θ for which the sequence $\{a_n\}$ converges. ($0 \leq \theta < \pi$)

Sol] Since the common ratio is $4\sin^2\theta + 2\cos\theta - 3$, $-1 < 4\sin^2\theta + 2\cos\theta - 3 < 1$

$$4\sin^2\theta + 2\cos\theta - 3 = 4(1 - \cos^2\theta) + 2\cos\theta - 3 \quad \leftarrow \sin^2\theta + \cos^2\theta = 1$$

$$= -4\cos^2\theta + 2\cos\theta + 1$$

Therefore, rearranging the inequality,

$$-1 < -4\cos^2\theta + 2\cos\theta + 1 < 1$$

So,

$$-1 < -4\cos^2\theta + 2\cos\theta + 1 \quad \dots \textcircled{1} \text{ and also } -4\cos^2\theta + 2\cos\theta + 1 < 1 \quad \dots \textcircled{2}$$

From $\textcircled{1}$, $2\cos^2\theta - \cos\theta - 1 < 0$; therefore, $(2\cos\theta + 1)(\cos\theta - 1) < 0$

$$\therefore -\frac{1}{2} < \cos\theta < 1 \quad \dots \textcircled{3}$$

From $\textcircled{2}$, $2\cos^2\theta - \cos\theta \geq 0$; therefore, $\cos\theta(2\cos\theta - 1) \geq 0$

$$\therefore \cos\theta \leq 0, \frac{1}{2} \leq \cos\theta \quad \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, $-\frac{1}{2} < \cos\theta \leq 0, \frac{1}{2} \leq \cos\theta < 1$

Since $0 \leq \theta < \pi$,

$$\text{when } \cos\theta = -\frac{1}{2}, \quad \theta = \frac{2}{3}\pi$$

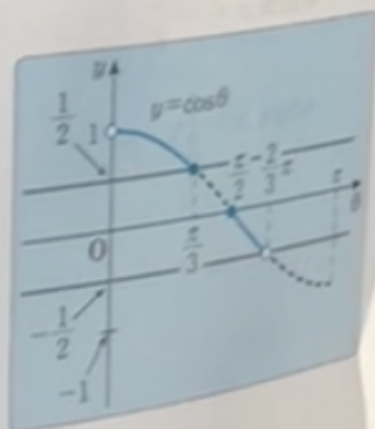
$$\text{when } \cos\theta = 0, \quad \theta = \frac{\pi}{2}$$

$$\text{when } \cos\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{3}$$

$$\text{when } \cos\theta = 1, \quad \theta = 0$$

Therefore, the range of values of θ is

$$0 < \theta < \frac{\pi}{3}, \quad \frac{\pi}{2} \leq \theta < \frac{2}{3}\pi$$



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Find the limits of each given sequence $\{a_n\}$ defined by the following condition**Ex.**

$$a_1 = 1, \quad a_{n+1} = \frac{2}{3}a_n - 1 \quad (n = 1, 2, 3, \dots)$$

[Sol] Since $a_{n+1} = \frac{2}{3}a_n - 1$, $a_{n+1} + 3 = \frac{2}{3}(a_n + 3)$ ←

Let $b_n = a_n + 3$.

$$\begin{cases} b_{n+1} = \frac{2}{3}b_n & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 = a_1 + 3 = 4 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n = 4\left(\frac{2}{3}\right)^{n-1}$ ←

$$\therefore a_n = b_n - 3 = 4\left(\frac{2}{3}\right)^{n-1} - 3$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[4\left(\frac{2}{3}\right)^{n-1} - 3 \right] = -3$$

Replacing both a_{n+1} and a_n with x , $x = \frac{2}{3}x - 1$
 $\therefore x = -3$ (N44)

$\{b_n\}$ is the geometric sequence with 1st term 4 and common ratio $\frac{2}{3}$.

(1) $a_1 = 3, \quad a_{n+1} = \frac{1}{2}a_n + 3 \quad (n = 1, 2, 3, \dots)$

[Sol] Since $a_{n+1} = \frac{1}{2}a_n + 3$, $a_{n+1} - 6 = \frac{1}{2}(a_n - 6)$ ←

Let $b_n = a_n - 6$.

$$\begin{cases} b_{n+1} = \frac{1}{2}b_n & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 = a_1 - 6 = -3 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n = -3\left(\frac{1}{2}\right)^{n-1}$

$$\therefore a_n = b_n + 6 = -3\left(\frac{1}{2}\right)^{n-1} + 6$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[-3\left(\frac{1}{2}\right)^{n-1} + 6 \right] = 6$$

Replacing both a_{n+1} and a_n with x , $x = \frac{1}{2}x + 3$
 $\therefore x = 6$

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$$a_1 = 2, \quad a_{n+1} = \frac{a_n}{a_n + 3} \quad (n = 1, 2, 3, \dots)$$

Since $a_{n+1} = \frac{a_n}{a_n + 3}$, $\frac{1}{a_{n+1}} = \frac{a_n + 3}{a_n} = \frac{a_n}{a_n} + \frac{3}{a_n} = 1 + \frac{3}{a_n}$

$$\text{Let } b_n = \frac{1}{a_n}.$$

Since $b_{n+1} = 3b_n + 1$, $b_{n+1} + \frac{1}{2} = 3\left(b_n + \frac{1}{2}\right)$ ← Replacing both b_{n+1} and b_n with $x, x = 2r+1$
 $\therefore x = \frac{1}{2}$

$$\text{Let } c_n = b_n + \frac{1}{2}.$$

$$\begin{cases} c_{n+1} = 3c_n & \text{--- ①} \\ c_1 = b_1 + \frac{1}{2} = \frac{1}{a_1} + \frac{1}{2} = 1 & \text{--- ②} \end{cases}$$

From ① and ②, $c_n = 3^{n-1}$

$$\therefore b_n = c_n - \frac{1}{2} = 3^{n-1} - \frac{1}{2}$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{3^{n-1} - \frac{1}{2}} = \frac{2}{2 \cdot 3^{n-1} - 1}$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2}{2 \cdot 3^{n-1} - 1} = 0$$

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1. Find the range of values of θ which satisfies $\lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \frac{1}{2}$.

$$\left(0 \leq \theta < \frac{\pi}{2}\right)$$

[Sol] (i) When $0 \leq \tan \theta < 1$, i.e. $0 \leq \theta < \frac{\pi}{4}$,

$$\lim_{n \rightarrow \infty} \tan^n \theta = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \frac{0 + 2}{2 \cdot 0 + 2} = 1$$

(ii) When $\tan \theta = 1$, i.e. $\theta = \frac{\pi}{4}$,

$$\lim_{n \rightarrow \infty} \tan^n \theta = 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \frac{1 + 2}{2 \cdot 1 + 2} = \frac{3}{4}$$

(iii) When $\tan \theta > 1$, i.e. $\frac{\pi}{4} < \theta < \frac{\pi}{2}$,

$$\lim_{n \rightarrow \infty} \frac{1}{\tan^n \theta} = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{\tan^n \theta}}{2 + \frac{2}{\tan^n \theta}} = \frac{1 + 2 \cdot 0}{2 + 2 \cdot 0} = \frac{1}{2}$$

From (i) ~ (iii), the range of values of θ is

$$\frac{\pi}{4} < \theta < \frac{\pi}{2}$$

Given the sequence $\{a_n\}$ which is defined by $a_1=0$, $a_2=1$ and $a_{n+2}=\frac{1}{4}(a_{n+1}+3a_n)$ ($n=1, 2, 3, \dots$), solve the following questions.

- 1) Let $b_n = a_{n+1} - a_n$ ($n=1, 2, 3, \dots$). Express b_n in terms of n .

Sol] Since $a_{n+2}=\frac{1}{4}(a_{n+1}+3a_n)$,

$$a_{n+2} - \frac{1}{4}a_{n+1} - \frac{3}{4}a_n = 0 \quad \dots \textcircled{1}$$

Rearranging $\textcircled{1}$,

$$a_{n+2} - a_{n+1} = -\frac{3}{4}(a_{n+1} - a_n)$$

Let $b_n = a_{n+1} - a_n$.

$$\begin{cases} b_{n+1} = -\frac{3}{4}b_n \quad \dots \textcircled{2} \end{cases}$$

$$\begin{cases} b_1 = a_2 - a_1 = 1 \quad \dots \textcircled{3} \end{cases}$$

From $\textcircled{2}$ and $\textcircled{3}$, $b_n = \left(-\frac{3}{4}\right)^{n-1}$

Replacing a_{n+2} , a_{n+1} , a_n with x^2 , x , 1 respectively,

$$x^2 - \frac{1}{4}x - \frac{3}{4} = 0$$

$$4x^2 - x - 3 = 0$$

$$(4x+3)(x-1) = 0$$

$$\therefore x = -\frac{3}{4}, 1$$

(N47)

$\{b_n\}$ is the geometric sequence with 1st term

$a_2 - a_1 = 1$ and common ratio $-\frac{3}{4}$.

- 2) Express a_n in terms of n .

Sol] From (1), since the general term of the sequence of differences of $\{a_n\}$ is

$$\left(-\frac{3}{4}\right)^{n-1},$$

when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} \left(-\frac{3}{4}\right)^{k-1} = 0 + \frac{1 - \left(-\frac{3}{4}\right)^{n-1}}{1 - \left(-\frac{3}{4}\right)} \\ &= \frac{4}{7} \left[1 - \left(-\frac{3}{4}\right)^{n-1} \right] \quad \dots \textcircled{4} \end{aligned}$$

Since $a_1=0$, $\textcircled{4}$ is also true when $n=1$.

$$\therefore a_n = \frac{4}{7} \left[1 - \left(-\frac{3}{4}\right)^{n-1} \right]$$

- 3) Find the limit value $\lim_{n \rightarrow \infty} a_n$.

Sol] From (2), $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{4}{7} \left[1 - \left(-\frac{3}{4}\right)^{n-1} \right] = \frac{4}{7}$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

(N32)

Infinite Geometric Sequences

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1. Find the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{2^{2n} - 3^n} = \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{4^n - 3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{4}\right)^n + \left(\frac{1}{2}\right)^n}{1 - \left(\frac{3}{4}\right)^n} = 0 \quad \Rightarrow N72$$

$$(2) \lim_{n \rightarrow \infty} \frac{4^n + 2^n}{3^n} = \lim_{n \rightarrow \infty} \frac{4^n \left[1 + \left(\frac{1}{2}\right)^n\right]}{3^n} = \lim_{n \rightarrow \infty} \left(\frac{4}{3}\right)^n \left[1 + \left(\frac{1}{2}\right)^n\right] = \infty \quad \Rightarrow N7$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ \lim_{n \rightarrow \infty} \frac{4^n + 2^n}{3^n} = \lim_{n \rightarrow \infty} \left[\left(\frac{4}{3}\right)^n + \left(\frac{2}{3}\right)^n \right] = \lim_{n \rightarrow \infty} \left(\frac{4}{3}\right)^n \left[1 + \left(\frac{1}{2}\right)^n\right] = \infty \end{array} \right]$$

$$(3) \lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} \quad \Rightarrow N$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \frac{0}{0+1} = 0$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

(iii) When $r = -1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \frac{-1}{1+1} = -\frac{1}{2}$$

(iv) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{r}}{1 + \left(\frac{1}{r}\right)^{2n}} = \frac{\frac{1}{r}}{1+0} = \frac{1}{r}$$

Find the range of values of x for which the sequence $\left(\frac{x+3}{4}\right)^n$ converges.
Then, state the limit values.

N76

[1] Since the common ratio is $\frac{x+3}{4}$, $-1 < \frac{x+3}{4} \leq 1$

$$\therefore -7 < x \leq 1$$

Also, the limit values are,

$$\text{when } -7 < x < 1, \lim_{n \rightarrow \infty} \left(\frac{x+3}{4}\right)^n = 0 \text{ and}$$

$$\text{when } x=1, \lim_{n \rightarrow \infty} \left(\frac{x+3}{4}\right)^n = 1$$

Find the limit of the sequence (a_n) defined by the following conditions.

N78

$$a_1 = 1, a_{n+1} = 3 - \frac{1}{2}a_n \quad (n=1, 2, 3, \dots)$$

[1] Since $a_{n+1} = 3 - \frac{1}{2}a_n$, $a_{n+1} - 2 = -\frac{1}{2}(a_n - 2)$

$$\text{Let } b_n = a_n - 2.$$

$$\begin{cases} b_{n+1} = -\frac{1}{2}b_n & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 = a_1 - 2 = -1 & \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, b_n = -\left(-\frac{1}{2}\right)^{n-1}$$

$$\therefore a_n = b_n + 2 = -\left(-\frac{1}{2}\right)^{n-1} + 2$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[-\left(-\frac{1}{2}\right)^{n-1} + 2 \right] = 2$$

Infinite Geometric Series

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Given an infinite sequence $\{a_n\}$, the expression $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is called the **infinite series**, where a_1 and a_n are called the **1st term** and the **n^{th} term** respectively.

Also, given an infinite sequence $\{a_n\}$, let S_n be the sum of the first n terms.

When the infinite sequence $\{S_n\}$ converges, the infinite series ① is said to converge.

When the infinite sequence $\{S_n\}$ diverges, the infinite series ① is said to diverge.

Likewise, $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ which is the infinite series that is derived from the infinite geometric sequence with 1st term a and common ratio r is called the **infinite geometric series** with 1st term a and common ratio r .

Determine whether the following infinite geometric series converges or diverges.

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots$$

[Sol] When $a=0$, all terms become 0; therefore, the infinite geometric series converges to 0.

When $a \neq 0$, let S_n be the sum of the first n terms.

$$\text{When } r \neq 1, S_n = \frac{a(1-r^n)}{1-r} \dots \textcircled{1}$$

$$\text{When } r = 1, S_n = na$$

$$(i) \text{ When } |r| < 1, \text{ since } \lim_{n \rightarrow \infty} r^n = 0, \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

Therefore, the infinite geometric series converges and the sum

$$\text{is } \boxed{\frac{a}{1-r}}.$$

(ii) When $r = 1$,

since $S_n = na$ and $a \neq 0$, the sequence $\{S_n\}$ diverges.

Therefore, the infinite geometric series diverges.

(iii) When $r \leq -1$ and $r > 1$,

since the sequence $\{r^n\}$ diverges,

from ①, the sequence $\{S_n\}$ also **diverges**.

Therefore, the infinite geometric series **diverges**.

When $r < -1$,
 $\{r^n\}$ oscillates
(no limit).
When $r > 1$,
 $\lim_{n \rightarrow \infty} r^n = \infty$.
Therefore,
either case,
diverges. (i)

Convergence and Divergence of an Infinite Geometric Series

Given an infinite geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$, the following is true.

When $a \neq 0$,

if $|r| < 1$, then the series converges and the sum is $\frac{a}{1-r}$,

if $|r| \geq 1$, then the series diverges.

When $a = 0$, the series converges and the sum is 0.

Determine whether each given infinite geometric series converges or diverges. If it converges, find the sum S .

Ex.

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

[Sol] The 1st term is 2 and the common ratio is $\frac{1}{2}$. Since $\left|\frac{1}{2}\right| < 1$, the series converges.

$$\therefore S = \frac{2}{1 - \frac{1}{2}} = 4$$

$$(1) \quad 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$$

[Sol] The 1st term is 1 and the common ratio is $\frac{1}{3}$. Since $\left|\frac{1}{3}\right| < 1$, the series converges.

$$\therefore S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

$$(2) \quad 3 - \frac{3}{10} + \frac{3}{100} - \frac{3}{1000} + \dots$$

[Sol] The 1st term is 3 and the common ratio is $-\frac{1}{10}$. Since $\left|-\frac{1}{10}\right| < 1$, the series converges.

$$\therefore S = \frac{3}{1 - \left(-\frac{1}{10}\right)} = \frac{30}{11}$$

$$(3) \quad -1 + 2 - 4 + 8 - \dots$$

The 1st term is -1 and the common ratio is -2 . Since $|-2| > 1$, the series diverges.

Infinite Geometric Series

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Determine whether each given infinite geometric series converges or diverges. If it converges, find the sum S .

$$(1) \quad 1 - \frac{\sqrt{2}}{2} + \frac{1}{2} - \frac{\sqrt{2}}{4} + \dots$$

[Sol] The 1st term is 1 and the common ratio is $-\frac{\sqrt{2}}{2}$. Since $\left| -\frac{\sqrt{2}}{2} \right| < 1$, the series **converges**.

$$\begin{aligned} \therefore S &= \frac{1}{1 - \left(-\frac{\sqrt{2}}{2} \right)} \\ &= \frac{2}{2 + \sqrt{2}} \\ &= \frac{2(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})} \\ &= 2 - \sqrt{2} \end{aligned}$$

Multiplying the numerator and the denominator by $2 - \sqrt{2}$

$$(2) \quad 6 - 2\sqrt{3} + 2 - \frac{2\sqrt{3}}{3} + \dots$$

[Sol] The 1st term is 6 and the common ratio is $\frac{-2\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}$.

Since $\left| -\frac{\sqrt{3}}{3} \right| < 1$, the series **converges**.

$$\begin{aligned} \therefore S &= \frac{6}{1 - \left(-\frac{\sqrt{3}}{3} \right)} \\ &= \frac{18}{3 + \sqrt{3}} \\ &= \frac{18(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})} \\ &= 3(3 - \sqrt{3}) \end{aligned}$$

32b

$$\sqrt{2} + (2 - \sqrt{2}) + (3\sqrt{2} - 4) + (10 - 7\sqrt{2}) + \dots$$

[Sol] The 1st term is $\sqrt{2}$ and the common ratio is $\frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1$.
 Since $|\sqrt{2}-1| < 1$, the series converges.

$$\begin{aligned}\therefore S &= \frac{\sqrt{2}}{1 - (\sqrt{2}-1)} \\ &= \frac{\sqrt{2}}{2-\sqrt{2}} \\ &= \frac{\sqrt{2}(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} \\ &= \sqrt{2} + 1\end{aligned}$$

$$(4) (\sqrt{3}-1) - 2(2-\sqrt{3}) + 2(3\sqrt{3}-5) - 4(7-4\sqrt{3}) + \dots - 2 =$$

[Sol] The 1st term is $\sqrt{3}-1$ and the common ratio is

$$\frac{-2(2-\sqrt{3})}{\sqrt{3}-1} = \frac{-2(2-\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = -\sqrt{3}+1$$

Since $|\sqrt{3}-1| < 1$, the series converges.

$$\begin{aligned}\therefore S &= \frac{\sqrt{3}-1}{1 - (-\sqrt{3}+1)} \\ &= \frac{\sqrt{3}-1}{\sqrt{3}} \\ &= \frac{3-\sqrt{3}}{3}\end{aligned}$$

Infinite Geometric Series

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Find the range of values of a real number x for which each given infinite geometric series converges. Then, find the sum S .

Ex. $1 + 2x + 4x^2 + \dots$

[Sol] Since the common ratio is $2x$, $-1 < 2x < 1$ ←

$$\therefore -\frac{1}{2} < x < \frac{1}{2}$$

Also, $S = \frac{1}{1-2x}$ ←

$$S = \frac{a}{1-r}$$

Let a be the 1st term and r be the common ratio. If $|r| < 1$ when $a \neq 0$, then the infinite geometric series converges.

(1) $3 + x + \frac{x^2}{3} + \dots$

[Sol] Since the common ratio is $\frac{x}{3}$, $-1 < \frac{x}{3} < 1$

$$\therefore -3 < x < 3$$

Also, $S = \frac{3}{1 - \frac{x}{3}} = \frac{9}{3-x}$

(2) $1 + (2-x^2) + (2-x^2)^2 + \dots$

[Sol] Since the common ratio is $2-x^2$, $-1 < 2-x^2 < 1$

So, $-1 < 2-x^2 \dots \textcircled{1}$ and also $2-x^2 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 < 3$; therefore, $-\sqrt{3} < x < \sqrt{3} \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 > 1$; therefore, $x < -1$, $1 < x \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$-\sqrt{3} < x < -1, \quad 1 < x < \sqrt{3}$$

Also, $S = \frac{1}{1 - (2-x^2)} = \frac{1}{x^2-1}$

3b

$$x + x(1-x^2) + x(1-x^2)^2 + \dots$$

(i) When $x \neq 0$, since the common ratio is $1-x^2$, $-1 < 1-x^2 < 1$

So, $-1 < 1-x^2 \dots \textcircled{1}$ and also $1-x^2 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 < \boxed{2}$; therefore, $-\sqrt{2} < x < \sqrt{2} \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 > \boxed{0}$; therefore, $x < \boxed{0}$, $\boxed{0} < x \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, $-\sqrt{2} < x < \boxed{0}$, $\boxed{0} < x < \sqrt{2}$

$$\text{Also, } S = \frac{\boxed{x}}{1-(1-x^2)} = \boxed{\frac{1}{x}}$$

(ii) When $x=0$, all terms become 0; therefore, the series converges.

$$\therefore S = \boxed{0}$$

From (i) and (ii), the range of values of x is $-\sqrt{2} < x < \sqrt{2}$.

Also, when $-\sqrt{2} < x < \boxed{0}$, $\boxed{0} < x < \sqrt{2}$, $S = \frac{1}{x}$ and

$$\text{when } x = \boxed{0}, S = \boxed{0}$$

$$x + x(1-x)^2 + x(1-x)^4 + \dots$$

(i) When $x \neq 0$, since the common ratio is $(1-x)^2$, $-1 < (1-x)^2 < 1$

So, $-1 < (1-x)^2 \dots \textcircled{1}$ and also $(1-x)^2 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $(1-x)^2 + 1 > 0$; therefore, it is true for all real numbers x .

From $\textcircled{2}$, $x(x-2) < 0$; therefore, $0 < x < 2 \dots \textcircled{3}$

From $\textcircled{1}$ and $\textcircled{3}$, $0 < x < 2$

$$\text{Also, } S = \frac{x}{1-(1-x)^2} = -\frac{1}{x-2}$$

(ii) When $x=0$, all terms become 0; therefore, the series converges.

$$\therefore S = 0$$

From (i) and (ii), the range of values of x is $0 \leq x < 2$.

Also, when $0 < x < 2$, $S = -\frac{1}{x-2}$ and

$$\text{when } x = 0, S = 0$$

Infinite Geometric Series

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Ex.

Given the infinite geometric series whose sum is 9 and 2nd term is -4, find the 1st term a and the common ratio r .

[Sol] Since the sum is 9, $a \neq 0$, $-1 < r < 1$ ←

Since the sum is 9, the infinite geometric series converges.

$$\begin{cases} \frac{a}{1-r} = 9 \quad \dots ① \\ ar = -4 \quad \dots ② \end{cases}$$

From ① and ②,

$$9(1-r) \cdot r = -4$$

$$9r^2 - 9r - 4 = 0$$

$$(3r+1)(3r-4) = 0$$

$$r = -\frac{1}{3}, \frac{4}{3}$$

Since $-1 < r < 1$, $r = -\frac{1}{3}$

From ②, $a = 12$

From ①, $a = 9(1-r)$
Substituting this into ②

1. Given the infinite geometric series whose sum is $\frac{8}{3}$ and 2nd term is -2, find the 1st term a and the common ratio r .

[Sol] Since the sum is $\frac{8}{3}$, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = \frac{8}{3} \quad \dots ① \\ ar = -2 \quad \dots ② \end{cases}$$

From ① and ②,

$$\frac{8}{3}(1-r) \cdot r = -2$$

$$4r^2 - 4r - 3 = 0$$

$$(2r+1)(2r-3) = 0$$

$$r = -\frac{1}{2}, \frac{3}{2}$$

Since $-1 < r < 1$, $r = -\frac{1}{2}$

From ②, $a = 4$

2. Given the infinite geometric series whose sum is 3, squaring each term of the series gives another infinite geometric series whose sum is 6. Find the 1st term a and the common ratio r of the original infinite geometric series.

[Sol] Since the sum is 3, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 3 & \dots \textcircled{1} \\ \frac{a^2}{1-r^2} = 6 & \dots \textcircled{2} \end{cases}$$

1st term: a^2 , common ratio: r^2 , sum: 6

From ① and ②,

$$\frac{[3(1-r)]^2}{1-r^2} = 6$$

From ①, $a = 3(1-r)$
Substituting this into ②

$$\frac{9(1-r)^2}{(1+r)(1-r)} = 6$$

$$3(1-r) = 2(1+r)$$

$$\therefore r = \frac{1}{5}$$

$$\text{From ①, } a = \frac{12}{5}$$

Alternative Solution

Since the sum is 3,
 $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 3 & \dots \textcircled{1} \\ \frac{a^2}{1-r^2} = 6 & \dots \textcircled{2} \end{cases}$$

From ①, $a = 3(1-r)$ $\dots \textcircled{3}$

From ②, $a^2 = 6(1-r^2)$ $\dots \textcircled{4}$

From ③ and ④,

$$[3(1-r)]^2 = 6(1-r^2)$$

$$5r^2 - 6r + 1 = 0$$

$$(3r-1)(r-1) = 0$$

$$r = \frac{1}{5}, 1$$

Since $-1 < r < 1$, $r = \frac{1}{5}$

$$\text{From ③, } a = \frac{12}{5}$$

3. Given the infinite geometric series whose sum is 2, cubing each term of the series gives another infinite geometric series whose sum is $\frac{32}{13}$. Find the 1st term a and the common ratio r of the original infinite geometric series.

[Sol] Since the sum is 2, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 2 & \dots \textcircled{1} \\ \frac{a^3}{1-r^3} = \frac{32}{13} & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$\frac{[2(1-r)]^3}{1-r^3} = \frac{32}{13}$$

$$\frac{8(1-r)^3}{(1-r)(1+r+r^2)} = \frac{32}{13}$$

$$13(1-r)^2 = 4(1+r+r^2)$$

$$3r^2 - 10r + 3 = 0$$

$$(3r-1)(r-3) = 0$$

$$r = \frac{1}{3}, 3$$

Since $-1 < r < 1$, $r = \frac{1}{3}$

$$\text{From ①, } a = \frac{4}{3}$$

Alternative Solution

Since the sum is 2,
 $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 2 & \dots \textcircled{1} \\ \frac{a^3}{1-r^3} = \frac{32}{13} & \dots \textcircled{2} \end{cases}$$

From ①, $a = 2(1-r)$ $\dots \textcircled{3}$

From ②, $a^3 = \frac{32}{13}(1-r^3)$ $\dots \textcircled{4}$

From ③ and ④,

$$[2(1-r)]^3 = \frac{32}{13}(1-r^3)$$

$$3r^3 - 13r^2 + 13r - 3 = 0$$

$$(r-1)(3r^2 - 10r + 3) = 0$$

$$(r-1)(3r-1)(r-3) = 0$$

$$r = \frac{1}{3}, 1, 3$$

Since $-1 < r < 1$, $r = \frac{1}{3}$

$$\text{From ③, } a = \frac{4}{3}$$

Infinite Geometric Series

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A decimal which contains a digit or block of digits that repeats without end in its decimal part is called a **repeating decimal**. The repeating decimal is expressed by placing “.” over the first and the last repeating digits as shown below:

$$0.33333\ldots = 0.\dot{3}, \quad 0.454545\ldots = 0.4\dot{5}, \quad 0.123123123\ldots = 0.\dot{1}2\dot{3}$$

Also, the repeating decimal can be expressed by a fraction using the sum of an infinite geometric series.

Convert the following repeating decimals into fractions.

Ex. $0.\dot{6}$

[Sol] $0.\dot{6} = 0.6 + 0.06 + 0.006 + \cdots$

$0.\dot{6}$ is the infinite geometric series with 1st term 0.6 and common ratio

0.1. Since $|0.1| < 1$, it converges.

$$\therefore 0.\dot{6} = \frac{0.6}{1-0.1} = \frac{2}{3}$$

(1) $0.\dot{1}2$

[Sol] $0.\dot{1}2 = 0.12 + 0.0012 + 0.000012 + \cdots$

$0.\dot{1}2$ is the infinite geometric series with 1st term 0.12 and common ratio 0

. Since $|0.01| < 1$, it converges.

$$\therefore 0.\dot{1}2 = \frac{0.12}{1-0.01} = \frac{4}{33}$$

(2) $0.\dot{3}0\dot{6}$

[Sol] $0.\dot{3}0\dot{6} = 0.306 + 0.000306 + 0.000000306 + \cdots$

$0.\dot{3}0\dot{6}$ is the infinite geometric series with 1st term 0.306 and common ratio

0.001. Since $|0.001| < 1$, it converges.

$$\therefore 0.\dot{3}0\dot{6} = \frac{0.306}{1-0.001} = \frac{34}{111}$$

185b

3) $0.3\dot{2}4$

[Sol] $0.3\dot{2}4 = 0.3 + 0.024 + 0.00024 + 0.0000024 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.024 and common ratio 0.01.

Since $|0.01| < 1$, it converges.

$$\therefore 0.3\dot{2}4 = 0.3 + \frac{0.024}{1 - 0.01} = \frac{3}{10} + \frac{4}{165} = \frac{107}{330}$$

(4) $3.5\dot{2}$

[Sol] $3.5\dot{2} = 3.5 + 0.02 + 0.002 + 0.0002 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.02 and common ratio 0.1.

Since $|0.1| < 1$, it converges.

$$\therefore 3.5\dot{2} = 3.5 + \frac{0.02}{1 - 0.1} = \frac{7}{2} + \frac{1}{45} = \frac{317}{90}$$

(5) $5.4\dot{7}7$

[Sol] $5.4\dot{7}7 = 5 + 0.477 + 0.000477 + 0.000000477 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.477 and common ratio 0.001.

Since $|0.001| < 1$, it converges.

$$\therefore 5.4\dot{7}7 = 5 + \frac{0.477}{1 - 0.001} = 5 + \frac{53}{111} = \frac{608}{111}$$

Infinite Geometric Series

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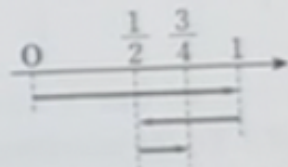
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Ex. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{2}$ units in the negative direction. Then, it moves $\frac{1}{2^2}$ units in the positive direction and then moves $\frac{1}{2^3}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

$$\begin{aligned}
 &1 \\
 &1 - \frac{1}{2} \\
 &1 - \frac{1}{2} + \frac{1}{2^2} \\
 &1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} \\
 &\vdots
 \end{aligned}$$



Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio

$$-\frac{1}{2}. \text{ Since } \left| -\frac{1}{2} \right| < 1, \text{ it converges.}$$

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{2} \right)} = \frac{2}{3}$$

Thus, the coordinate of the point which point P approaches

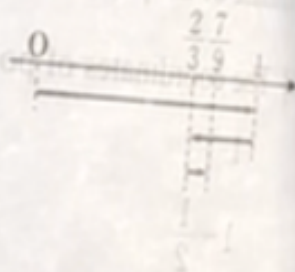
$$\text{is } \frac{2}{3}.$$

N86b

1. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{3}$ units in the negative direction. Then, it moves $\frac{1}{3^2}$ units in the positive direction and then moves $\frac{1}{3^3}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

$$\begin{aligned} &1 \\ &1 - \frac{1}{3} \\ &1 - \frac{1}{3} + \frac{1}{3^2} \\ &1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} \\ &\vdots \end{aligned}$$



Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio $-\frac{1}{3}$.

Since $\left| -\frac{1}{3} \right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{3} \right)} = \frac{3}{4}$$

Thus, the coordinate of the point which point P approaches is $\frac{3}{4}$.

Infinite Geometric Series

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Ex.

On a coordinate plane, starting from origin O, point P moves 1 unit in the positive direction to the x -axis and then moves $\frac{1}{2}$ units in the positive direction to the y -axis. Then, it moves $\frac{1}{2^2}$ units in the negative direction to the x -axis and then moves $\frac{1}{2^3}$ units in the negative direction to the y -axis. When this motion continues infinitely, find the coordinates of the point which point P approaches.

[Sol] Let (x, y) be the coordinates of the point which point P approaches.

The x -coordinate of point P is

$$x = 1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio

$-\frac{1}{4}$. Since $\left| -\frac{1}{4} \right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{4} \right)} = \frac{4}{5}$$

The y -coordinate of point P is

$$y = \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2^5} - \frac{1}{2^7} + \dots$$

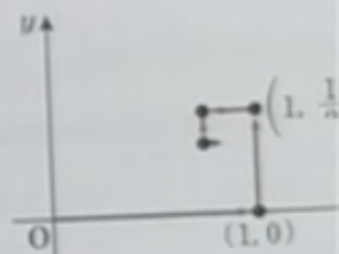
y is the infinite geometric series with 1st term $\frac{1}{2}$ and common ratio

$-\frac{1}{4}$. Since $\left| -\frac{1}{4} \right| < 1$, it converges.

$$\therefore y = \frac{\frac{1}{2}}{1 - \left(-\frac{1}{4} \right)} = \frac{2}{5}$$

Thus, the coordinates of the point which point P approaches

are $\left(\frac{4}{5}, \frac{2}{5} \right)$.



N87b

1. On a coordinate plane, starting from origin O , point P moves 1 unit in the positive direction to the x -axis and then moves $\frac{2}{3}$ units in the positive direction to the y -axis. Then, it moves $\left(\frac{2}{3}\right)^2$ units in the positive direction to the x -axis and then moves $\left(\frac{2}{3}\right)^3$ units in the positive direction to the y -axis. When this motion continues infinitely, find the coordinates of the point which point P approaches.

[Sol] Let (x, y) be the coordinates of the point which point P approaches.

The x -coordinate of point P is

$$x = 1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^6 + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio $\frac{4}{9}$.

Since $\left|\frac{4}{9}\right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \frac{4}{9}} = \frac{9}{5}$$

The y -coordinate of point P is

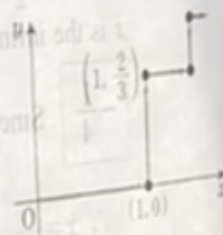
$$y = \frac{2}{3} + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^7 + \dots$$

y is the infinite geometric series with 1st term $\frac{2}{3}$ and common ratio $\frac{4}{9}$.

Since $\left|\frac{4}{9}\right| < 1$, it converges.

$$\therefore y = \frac{\frac{2}{3}}{1 - \frac{4}{9}} = \frac{6}{5}$$

Thus, the coordinates of the point which point P approaches are $\left(\frac{9}{5}, \frac{6}{5}\right)$.



Infinite Geometric Series

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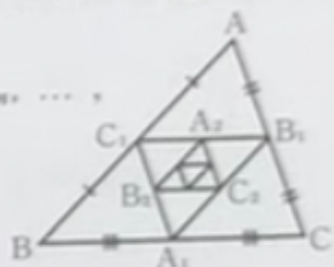
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Ex.

Given that $\triangle ABC$ has a perimeter of 3, form $\triangle A_1B_1C_1$ whose vertices are the midpoints of the sides of $\triangle ABC$ and then form $\triangle A_2B_2C_2$ whose vertices are the midpoints of the sides of $\triangle A_1B_1C_1$. When triangles are formed infinitely as

$\triangle A_1B_1C_1, \triangle A_2B_2C_2, \triangle A_3B_3C_3, \dots, \triangle A_nB_nC_n, \dots$, find the sum L of the perimeters.

[Sol] Let L_n be the perimeter of $\triangle A_nB_nC_n$.



$$\begin{cases} L_{n+1} = \frac{1}{2}L_n & \dots \textcircled{1} \\ L_1 = \frac{1}{2} \cdot 3 = \frac{3}{2} & \dots \textcircled{2} \end{cases}$$

From the Midpoint Theorem (*),

$$A_{n+1}B_{n+1} = \frac{1}{2}A_nB_n, B_{n+1}C_{n+1} = \frac{1}{2}B_nC_n, C_{n+1}A_{n+1} = \frac{1}{2}C_nA_n$$

Therefore,

$$A_{n+1}B_{n+1} + B_{n+1}C_{n+1} + C_{n+1}A_{n+1} = \frac{1}{2}(A_nB_n + B_nC_n + C_nA_n)$$

From ① and ②,

$$L = \frac{3}{2} + \frac{3}{2} \cdot \frac{1}{2} + \frac{3}{2} \left(\frac{1}{2}\right)^2 + \frac{3}{2} \left(\frac{1}{2}\right)^3 + \dots$$

L is the infinite geometric series with 1st term $\frac{3}{2}$ and common ratio

$\frac{1}{2}$. Since $\left|\frac{1}{2}\right| < 1$, it converges.

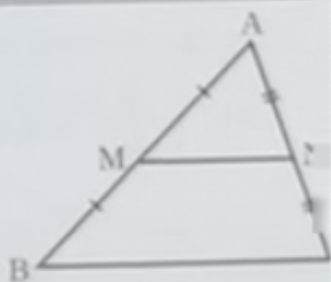
$$\therefore L = \frac{\frac{3}{2}}{1 - \frac{1}{2}} = 3$$

ANS: $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \dots$

* If M and N are the midpoints of two sides AB and AC of $\triangle ABC$, the following relationships are true:

MN and BC are parallel, $MN = \frac{1}{2}BC$

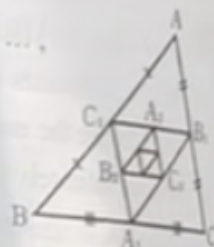
This is called the *Midpoint Theorem*.



N88b

1. Given that $\triangle ABC$ has a perimeter of 10, form $\triangle A_1B_1C_1$ whose vertices are the midpoints of the sides of $\triangle ABC$ and then form $\triangle A_2B_2C_2$ whose vertices are the midpoints of the sides of $\triangle A_1B_1C_1$. When triangles are formed infinitely as

$\triangle A_1B_1C_1, \triangle A_2B_2C_2, \triangle A_3B_3C_3, \dots, \triangle A_nB_nC_n, \dots$, find the sum L of the perimeters.



[Sol] Let L_n be the perimeter of $\triangle A_nB_nC_n$.

$$\begin{cases} L_{n+1} = \frac{1}{2}L_n & \dots \textcircled{1} \\ L_1 = \frac{1}{2} \cdot 10 = 5 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$L = 5 + 5 \cdot \frac{1}{2} + 5 \left(\frac{1}{2}\right)^2 + 5 \left(\frac{1}{2}\right)^3 + \dots$$

L is the infinite geometric series with 1st term 5 and common ratio $\frac{1}{2}$.

Since $\left|\frac{1}{2}\right| < 1$, it converges.

$$\therefore L = \frac{5}{1 - \frac{1}{2}} = 10$$

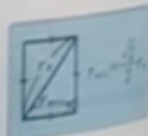
2. Given square S_n and circle C_n ($n=1, 2, \dots$), C_n is inscribed in S_n and S_{n+1} is inscribed in C_n . Let r_n be the radius of C_n and a be the length of each side of S_1 . Find the sum L of the circumferences.



Circumference = $2 \times \text{radius} \times \pi$,
therefore, $L_n = 2\pi r_n$.

[Sol] Let L_n be the circumference of C_n .

$$\begin{cases} L_{n+1} = 2\pi r_{n+1} = 2\pi \cdot \frac{\sqrt{2}}{2} r_n = \frac{\sqrt{2}}{2} L_n & \dots \textcircled{1} \\ L_1 = 2\pi r_1 = \pi a & \dots \textcircled{2} \end{cases}$$



The diameter of C_1 = the length of each side of S_1 ,
therefore, $2r_1 = a$.

From ① and ②,

$$L = \pi a + \pi a \cdot \frac{\sqrt{2}}{2} + \pi a \left(\frac{\sqrt{2}}{2}\right)^2 + \dots$$

L is the infinite geometric series with 1st term πa and common ratio $\frac{\sqrt{2}}{2}$.

Since $\left|\frac{\sqrt{2}}{2}\right| < 1$, it converges.

$$\therefore L = \frac{\pi a}{1 - \frac{\sqrt{2}}{2}} = \frac{2\pi a}{2 - \sqrt{2}} = \frac{2\pi a(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = (2 + \sqrt{2})\pi a$$

Infinite Geometric Series

Name _____

Date ____/____/____

Time ____:____ to ____:____

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1. Let n be a positive integer and r be a real number that satisfies $r > 1$.
Given that the sequence (a_n) satisfies $a_1r + a_2r^2 + \dots + a_nr^n = (-1)^n$,
solve the following questions.

- (1) Express a_1, a_2, a_3 in terms of r .

[Sol] When $n=1$, $a_1r = -1$; therefore, $a_1 = -\frac{1}{r}$

When $n=2$, $a_1r + a_2r^2 = 1$; therefore, $a_2 = \frac{2}{r^2}$ ← $-1 + a_2r^2 = 1$

When $n=3$, $a_1r + a_2r^2 + a_3r^3 = -1$; therefore, $a_3 = -\frac{2}{r^3}$ ← $1 + a_3r^3 = -1$

- (2) When $n \geq 2$, express a_n in terms of r .

[Sol] When $n \geq 2$,

$$\begin{cases} a_1r + a_2r^2 + \dots + a_{n-1}r^{n-1} = (-1)^{n-1} & \dots \textcircled{1} \\ a_1r + a_2r^2 + \dots + a_{n-1}r^{n-1} + a_nr^n = (-1)^n & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$a_nr^n = (-1)^n - (-1)^{n-1}$$

$$\therefore a_n = \frac{(-1)^n - (-1)^{n-1}}{r^n} = 2\left(-\frac{1}{r}\right)^n$$

$$\begin{aligned} & \frac{(-1)^n - (-1)^{n-1}}{r^n} \\ &= \frac{(-1)^n + (-1)}{r^n} \\ &= \frac{2(-1)^n}{r^n} \end{aligned}$$

- (3) Find the sum of the infinite series $a_1 + a_2 + \dots + a_n + \dots$.

[Sol] From (1) and (2), $a_2 + a_3 + \dots + a_n + \dots$ is the infinite geometric series
with 1st term $\frac{2}{r^2}$ and common ratio $-\frac{1}{r}$.

Since $\left| -\frac{1}{r} \right| < 1$, it converges.

$$\therefore a_1 + a_2 + \dots + a_n + \dots$$

$$= a_1 + (a_2 + a_3 + \dots + a_n + \dots)$$

$$= -\frac{1}{r} + \frac{\frac{2}{r^2}}{1 - \left(-\frac{1}{r}\right)}$$

$$= -\frac{r-1}{r(r+1)}$$

From (1),
 $a_1 = -\frac{1}{r}$

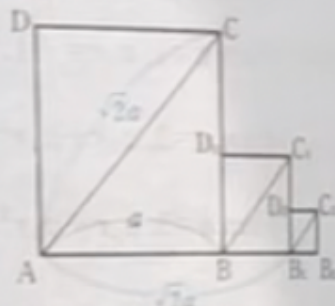
N89b

2. Given square ABCD with side length a , extend the line of side AB through point B in the opposite direction to point A. Then, place point B_1 on this extended line of side AB such that $AB_1 = AC$. Make a square $BB_1C_1D_1$ using BB_1 as a side and extend the line of side BB_1 from point B_1 in the opposite direction to point B. Then, place point B_2 on this extended line of side BB_1 such that $BB_2 = BC_1$. Given points $B_1, B_2, B_3, \dots$ as described above, find the sum of the lengths of line segments $AB, BB_1, B_1B_2, \dots$.

[Sol] Since $AB = a$,

$$\begin{aligned} BB_1 &= AB_1 - AB \\ &= \sqrt{2}AB - AB \\ &= \sqrt{2}a - a \\ &= (\sqrt{2} - 1)a \end{aligned}$$

$$\begin{aligned} AB_1 &= AC \\ &= \sqrt{2}AB \end{aligned}$$



$$\begin{aligned} B_1B_2 &= BB_2 - BB_1 \\ &= \sqrt{2}BB_1 - BB_1 \\ &= (\sqrt{2} - 1) \cdot (\sqrt{2} - 1)a \\ &= (\sqrt{2} - 1)^2 a \end{aligned}$$

$$\begin{aligned} B_2B_3 &= B_1B_3 - B_1B_2 \\ &= \sqrt{2}B_1B_2 - B_1B_2 \\ &= (\sqrt{2} - 1) \cdot (\sqrt{2} - 1)^2 a \\ &= (\sqrt{2} - 1)^3 a \\ &\vdots \end{aligned}$$

Let L be the sum of the lengths of line segments $AB, BB_1, B_1B_2, \dots$

$$L = a + (\sqrt{2} - 1)a + (\sqrt{2} - 1)^2 a + (\sqrt{2} - 1)^3 a + \dots$$

Therefore, L is the infinite geometric series with 1st term a and common ratio $\sqrt{2} - 1$. Since $|\sqrt{2} - 1| < 1$, it converges.

$$\therefore L = \frac{a}{1 - (\sqrt{2} - 1)} = \frac{a}{2 - \sqrt{2}} = \frac{a(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = \frac{2 + \sqrt{2}}{2} a$$

Thus, the sum of the lengths of line segments $AB, BB_1, B_1B_2, \dots$

$$\text{is } \frac{2 + \sqrt{2}}{2} a.$$

Infinite Geometric Series

Name _____

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1. Determine whether the following infinite geometric series converges or diverges. If it converges, find the sum S . ➡ N8

$$16 + 8\sqrt{2} + 8 + 4\sqrt{2} + \dots$$

[Sol] The 1st term is 16 and the common ratio is $\frac{8\sqrt{2}}{16} = \frac{\sqrt{2}}{2}$. Since $\left| \frac{\sqrt{2}}{2} \right| < 1$, the series **converges**.

$$\begin{aligned} \therefore S &= \frac{16}{1 - \frac{\sqrt{2}}{2}} \\ &= \frac{32}{2 - \sqrt{2}} \\ &= \frac{32(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} \\ &= 16(2 + \sqrt{2}) \end{aligned}$$

2. Find the range of values of a real number x for which the following infinite geometric series converges. Then, find the sum S . ➡ N

$$2 + 2(x^2 - 3) + 2(x^2 - 3)^2 + \dots$$

[Sol] Since the common ratio is $x^2 - 3$, $-1 < x^2 - 3 < 1$

So, $-1 < x^2 - 3 \dots \textcircled{1}$ and also $x^2 - 3 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 > 2$; therefore, $x < -\sqrt{2}$, $\sqrt{2} < x \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 < 4$; therefore, $-2 < x < 2 \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$-2 < x < -\sqrt{2}, \quad \sqrt{2} < x < 2$$

$$\text{Also, } S = \frac{2}{1 - (x^2 - 3)} = \frac{2}{4 - x^2}$$

N90b

3. Convert the following repeating decimal into a fraction.
 $0.03\overline{18}$

[Sol] $0.03\overline{18} = 0.03 + 0.0018 + 0.000018 + 0.00000018 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.0018 and common ratio 0.01.

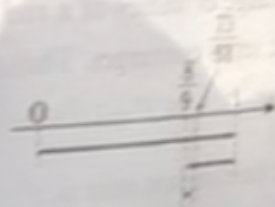
Since $|0.01| < 1$, it converges.

$$\therefore 0.03\overline{18} = 0.03 + \frac{0.0018}{1 - 0.01} = \frac{3}{100} + \frac{1}{500} = \frac{7}{220}$$

4. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{3^2}$ units in the negative direction. Then, it moves $\frac{1}{3^4}$ units in the positive direction and then moves $\frac{1}{3^6}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

$$\begin{aligned} &1 \\ &1 - \frac{1}{3^2} \\ &1 - \frac{1}{3^2} + \frac{1}{3^4} \\ &1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} \\ &\vdots \end{aligned}$$



Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio $-\frac{1}{9}$.

Since $\left| -\frac{1}{9} \right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{9} \right)} = \frac{9}{10}$$

the coordinate of the point which point P approaches is $\frac{9}{10}$.

Infinite Series

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Given the infinite series $a_1 + a_2 + a_3 + \dots + a_n + \dots$ ①, the sum of the first n terms

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

is called the **partial sum** of the first n terms of the infinite series.

The infinite series ① can be written as $\sum_{n=1}^{\infty} a_n$.

Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\begin{aligned} \text{[Sol]} \quad \frac{1}{\sqrt{n+1} + \sqrt{n}} &= \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})} \\ &= \sqrt{n+1} - \sqrt{n} \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \dots + (\sqrt{n+1} - \sqrt{n}) \\ &= -1 + \sqrt{n+1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (-1 + \sqrt{n+1}) = \infty$$

Thus, the series diverges.

$$(1) \quad \sum_{n=1}^{\infty} \frac{3}{\sqrt{3n+2} + \sqrt{3n-1}}$$

$$\begin{aligned} \text{[Sol]} \quad \frac{3}{\sqrt{3n+2} + \sqrt{3n-1}} &= \frac{3(\sqrt{3n+2} - \sqrt{3n-1})}{(\sqrt{3n+2} + \sqrt{3n-1})(\sqrt{3n+2} - \sqrt{3n-1})} \\ &= \sqrt{3n+2} - \sqrt{3n-1} \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= (\sqrt{5} - \sqrt{2}) + (\sqrt{8} - \sqrt{5}) + (\sqrt{11} - \sqrt{8}) + \dots + (\sqrt{3n+2} - \sqrt{3n-1}) \\ &= -\sqrt{2} + \sqrt{3n+2} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (-\sqrt{2} + \sqrt{3n+2}) = \infty$$

Thus, the series diverges.

N91b

$$(2) \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1} + \sqrt{2n-1}}$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{\sqrt{2n+1} + \sqrt{2n-1}} &= \frac{\sqrt{2n+1} - \sqrt{2n-1}}{(\sqrt{2n+1} + \sqrt{2n-1})(\sqrt{2n+1} - \sqrt{2n-1})} \\ &= \frac{1}{2}(\sqrt{2n+1} - \sqrt{2n-1}) \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \frac{1}{2}(\sqrt{3} - \sqrt{1}) + \frac{1}{2}(\sqrt{5} - \sqrt{3}) + \frac{1}{2}(\sqrt{7} - \sqrt{5}) + \cdots + \frac{1}{2}(\sqrt{2n+1} - \sqrt{2n-1}) \\ &= \frac{1}{2}(-1 + \sqrt{2n+1}) \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2}(-1 + \sqrt{2n+1}) = \infty$$

Thus, the series diverges.

$$(3) \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} &= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{[(n+1)\sqrt{n} + n\sqrt{n+1}][(n+1)\sqrt{n} - n\sqrt{n+1}]} \\ &= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n(n+1)} \\ &= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \\ &= 1 - \frac{1}{\sqrt{n+1}} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$$

Thus, the series converges and the sum is 1.

Infinite Series

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$

[Sol] Let $\frac{1}{n(n+1)} = \frac{a}{n} - \frac{b}{n+1}$.

$$1 = a(n+1) - bn$$

$$= (a-b)n + a$$

$$\therefore a=1, b=1$$

Multiplying both sides
by $n(n+1)$

Since $1 = (a-b)n + a$,

$$\begin{cases} a-b=0 \\ a=1 \end{cases}$$

$$a=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Thus, the series converges and the sum is 1.

(1) $\frac{3}{1 \cdot 4} + \frac{3}{4 \cdot 7} + \frac{3}{7 \cdot 10} + \cdots + \frac{3}{(3n-2)(3n+1)} + \cdots$

[Sol] Let $\frac{3}{(3n-2)(3n+1)} = \frac{a}{3n-2} - \frac{b}{3n+1}$.

$$3 = a(3n+1) - b(3n-2)$$

$$= 3(a-b)n + (a+2b)$$

$$\therefore a=1, b=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$$

$$= 1 - \frac{1}{3n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n+1}\right) = 1$$

Thus, the series converges and the sum is 1.

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$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)} + \dots$$

sol] Let $\frac{1}{(2n-1)(2n+1)} = \frac{a}{2n-1} - \frac{b}{2n+1}$

$$1 = a(2n+1) - b(2n-1)$$

$$= 2(a-b)n + (a+b)$$

$$\therefore a = \frac{1}{2}, b = \frac{1}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \frac{1}{2} \left(\frac{1}{5} - \frac{1}{7} \right) + \dots + \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right)$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) = \frac{1}{2}$$

Thus, the series converges and the sum is $\frac{1}{2}$.

(3) $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+\dots+n} + \dots$

[Sol] $\frac{1}{1+2+\dots+n} = \frac{1}{\frac{1}{2}n(n+1)} = \frac{2}{n(n+1)}$

Let $\frac{2}{n(n+1)} = \frac{a}{n} - \frac{b}{n+1}$

$$2 = a(n+1) - bn$$

$$= (a-b)n + a$$

$$\therefore a = 2, b = 2$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = 2 \left(\frac{1}{1} - \frac{1}{2} \right) + 2 \left(\frac{1}{2} - \frac{1}{3} \right) + 2 \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + 2 \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 2 \left(1 - \frac{1}{n+1} \right)$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n+1} \right) = 2$$

Thus, the series converges and the sum is 2.

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

$$(1) \quad \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots$$

[Sol] Let $\frac{1}{n(n+1)(n+2)} = \frac{a}{n(n+1)} - \frac{b}{(n+1)(n+2)}$.

$$1 = a(n+2) - bn$$

$$= (a-b)n + 2a$$

$$\therefore a = \frac{1}{2}, \quad b = \frac{1}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{6} \right) + \frac{1}{2} \left(\frac{1}{6} - \frac{1}{12} \right) + \frac{1}{2} \left(\frac{1}{12} - \frac{1}{20} \right) + \cdots + \frac{1}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right] = \frac{1}{4}$$

Thus, the series converges and the sum is $\frac{1}{4}$.

N93b

$$(2) \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 + 2 \cdot 3} + \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4} + \dots + \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)} + \dots$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)} &= \frac{1}{\sum_{k=1}^n k(k+1)} \\ &= \frac{1}{\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)} \\ &= \frac{1}{\frac{1}{6}n(n+1)[(2n+1)+3]} \\ &= \frac{3}{n(n+1)(n+2)} \end{aligned}$$

$$\text{Let } \frac{3}{n(n+1)(n+2)} = \frac{a}{n(n+1)} - \frac{b}{(n+1)(n+2)}, \quad \frac{1}{n} = \frac{1}{n} \cdot 1$$

$$3 = a(n+2) - bn$$

$$= (a-b)n + 2a$$

$$\therefore a = \frac{3}{2}, \quad b = \frac{3}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{6} \right) + \frac{3}{2} \left(\frac{1}{6} - \frac{1}{12} \right) + \frac{3}{2} \left(\frac{1}{12} - \frac{1}{20} \right) + \dots + \frac{3}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right]$$

$$= \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right] = \frac{3}{4}$$

Thus, the series converges and the sum is $\frac{3}{4}$.

$$= \frac{1}{6}n(n+1)(2n+1)$$

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Find the sum of the following infinite series.

Ex. $\sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right)$

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n = \left(\cos \pi - \cos \frac{\pi}{2} \right) + \left(\cos \frac{\pi}{2} - \cos \frac{\pi}{3} \right) + \cdots + \left(\cos \frac{\pi}{n} - \cos \frac{\pi}{n+1} \right)$$

$$= \cos \pi - \cos \frac{\pi}{n+1}$$

$$= -1 - \cos \frac{\pi}{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(-1 - \cos \frac{\pi}{n+1} \right) = -2$$

$$\therefore \sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right) = -2$$

$$\lim_{n \rightarrow \infty} \cos \frac{\pi}{n+1} = \cos 0 = 1$$

(1). $\sum_{k=1}^{\infty} \left(\sin \frac{\pi}{2k} - \sin \frac{\pi}{2k+2} \right)$

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n = \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{4} \right) + \left(\sin \frac{\pi}{4} - \sin \frac{\pi}{6} \right) + \cdots + \left(\sin \frac{\pi}{2n} - \sin \frac{\pi}{2n+2} \right)$$

$$= \sin \frac{\pi}{2} - \sin \frac{\pi}{2n+2}$$

$$= 1 - \sin \frac{\pi}{2n+2}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \sin \frac{\pi}{2n+2} \right) = 1$$

$$\therefore \sum_{k=1}^{\infty} \left(\sin \frac{\pi}{2k} - \sin \frac{\pi}{2k+2} \right) = 1$$

N94b

$$(2) \sum_{k=1}^{\infty} 2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}}$$

$$[\text{Sol}] \quad 2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}}$$

$$= 2 \cdot \frac{1}{2} \left[\sin \left(\frac{3\pi}{2^{k+1}} + \frac{\pi}{2^{k+1}} \right) - \sin \left(\frac{3\pi}{2^{k+1}} - \frac{\pi}{2^{k+1}} \right) \right]$$

$$= \sin \frac{\pi}{2^{k-1}} - \sin \frac{\pi}{2^k}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\sin \pi - \sin \frac{\pi}{2} \right) + \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{4} \right) + \dots + \left(\sin \frac{\pi}{2^{n-1}} - \sin \frac{\pi}{2^n} \right)$$

$$= \sin \pi - \sin \frac{\pi}{2^n}$$

$$= -\sin \frac{\pi}{2^n}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(-\sin \frac{\pi}{2^n} \right) = 0$$

$$\therefore \sum_{k=1}^{\infty} 2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}} = 0$$

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Properties of Infinite Series

When the infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, where $\sum_{n=1}^{\infty} a_n = S$ and

$\sum_{n=1}^{\infty} b_n = T$, the following properties are true:

$$\sum_{n=1}^{\infty} k a_n = k S$$

(k is a constant)

$$\sum_{n=1}^{\infty} (a_n + b_n) = S + T$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = S - T$$

Find the sum of the following infinite series.

Ex.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n} \right)$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1 - \frac{1}{2}} = 2 \quad \leftarrow$$

Since the series is the infinite geometric series with 1st term 1 and common ratio $\frac{1}{2}$, it converges.

$$\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{\frac{5}{4}}{1 - \frac{1}{4}} = \frac{5}{3} \quad \leftarrow$$

Since the series is the infinite geometric series with 1st term $\frac{5}{4}$ and common ratio $\frac{1}{4}$, it converges.

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{5}{4^n} = 2 - \frac{5}{3} = \frac{1}{3}$$

$$(1) \sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}} \right)$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{\frac{2}{3}}{1 - \frac{1}{3}} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = \frac{1}{1 - \frac{1}{6}} = \frac{6}{5}$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}} \right) = \sum_{n=1}^{\infty} \frac{2}{3^n} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = 1 - \frac{6}{5} = -\frac{1}{5}$$

195b

$$2) \sum_{n=1}^{\infty} \frac{2^n - 3^n}{4^n}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{2^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n - 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = 1 - 3 = -2$$

$$(3) \sum_{n=1}^{\infty} \frac{3^{n+1} + 4^n}{5^n}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n} = \sum_{n=1}^{\infty} 3 \left(\frac{3}{5}\right)^n = \frac{\frac{9}{5}}{1 - \frac{3}{5}} = \frac{9}{2}$$

$$\sum_{n=1}^{\infty} \frac{4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \frac{\frac{4}{5}}{1 - \frac{4}{5}} = 4$$

$$\therefore \sum_{n=1}^{\infty} \frac{3^{n+1} + 4^n}{5^n} = \sum_{n=1}^{\infty} 3 \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \frac{9}{2} + 4 = \frac{17}{2}$$

$$(4) \sum_{n=1}^{\infty} \frac{3^{-n} - \left(\frac{1}{2}\right)^n}{2^{3n}}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{3^{-n}}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(3^{-1})^n}{8^n} = \sum_{n=1}^{\infty} \left(\frac{1}{24}\right)^n = \frac{\frac{1}{24}}{1 - \frac{1}{24}} = \frac{1}{23}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{8^n} = \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right)^n = \frac{-\frac{1}{8}}{1 - \left(-\frac{1}{8}\right)} = -\frac{1}{9}$$

$$\therefore \sum_{n=1}^{\infty} \frac{3^{-n} - (-1)^n}{2^{3n}} = \sum_{n=1}^{\infty} \left(\frac{1}{24}\right)^n - \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right)^n = \frac{1}{23} - \frac{1}{9} = \frac{32}{207}$$

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Let S_n be the partial sum of the first n terms of infinite series $\sum_{n=1}^{\infty} a_n$.

When $n \geq 2$, $a_n = S_n - S_{n-1}$

When the infinite series $\sum_{n=1}^{\infty} a_n$ converges, let S be the sum. Then,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0$$

Therefore, when $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$

Also, when the sequence $\{a_n\}$ does not converge to 0, $\sum_{n=1}^{\infty} a_n$ diverges.

From the above, the following statement is true.

Convergence and Divergence of Infinite Series

If the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

If the sequence $\{a_n\}$ does not converge to 0, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Determine whether each given infinite series converges or diverges.

Ex. $\frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots$

[Sol] The n^{th} term a_n is $a_n = \frac{n}{2n-1}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series diverges.

(1) $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \dots$

[Sol] The n^{th} term a_n is $a_n = \frac{n}{n+2}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = 1$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series diverges.

$$(2) \quad \frac{1}{3} + \frac{3}{6} + \frac{5}{9} + \frac{7}{12} + \dots$$

[Sol] The n^{th} term a_n is $a_n = \frac{2n-1}{3n}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{3} = \frac{2}{3}$$

The sequence $\{a_n\}$ does not converge to 0.

Therefore, the series **diverges**.

$$(3) \quad \frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \frac{16}{17} + \dots$$

[Sol] The n^{th} term a_n is $a_n = \frac{n^2}{n^2+1}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = 1$$

The sequence $\{a_n\}$ does not converge to 0.

Therefore, the series **diverges**.

$$(4) \quad 1 - 2 + 3 - 4 + \dots$$

[Sol] The n^{th} term a_n is $a_n = (-1)^{n-1}n$.

The sequence $\{a_n\}$ oscillates and does not converge to 0.

Therefore, the series **diverges**.

$$(5) \quad -1 + 3 - 5 + 7 - \dots$$

[Sol] The n^{th} term a_n is $a_n = (-1)^n(2n-1)$.

The sequence $\{a_n\}$ oscillates and does not converge to 0.

Therefore, the series **diverges**.

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\frac{1}{2} - \frac{2}{3} + \frac{2}{3} - \frac{3}{4} + \frac{3}{4} - \dots - \frac{n}{n+1} + \frac{n}{n+1} - \frac{n+1}{n+2} + \dots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$\begin{aligned} S_n &= S_{2m-1} \\ &= \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right) + \dots + \left(-\frac{m}{m+1} + \frac{m}{m+1}\right) \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \boxed{\frac{1}{2}}$$

$$S_1 = \frac{1}{2}$$

$$S_3 = \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right)$$

$$S_5 = \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right)$$

⋮

(ii) When $n = 2m$,

$$\begin{aligned} S_n &= S_{2m} \\ &= S_{2m-1} + \left(-\frac{m+1}{m+2}\right) \end{aligned}$$

$$S_{2m} = S_{2m-1} + (\text{the } 2m^{\text{th}} \text{ term})$$

$$\text{From (i), } \lim_{m \rightarrow \infty} S_{2m-1} = \frac{1}{2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{1 + \frac{1}{m}}{1 + \frac{2}{m}} \right) = \boxed{-\frac{1}{2}}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series diverges.

$\{S_n\}$ oscillates and diverges.

$$\frac{2}{1} - \frac{2}{1} + \frac{2}{1} - \frac{2}{1} + \dots$$

N97b

$$(1) \quad 3 - 2 + 2 - \frac{5}{3} + \frac{5}{3} - \dots - \frac{n+2}{n} + \frac{n+2}{n} - \frac{n+3}{n+1} + \dots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$S_n = S_{2m-1}$$

$$= 3 + (-2 + 2) + \left(-\frac{5}{3} + \frac{5}{3}\right) + \dots + \left(-\frac{m+2}{m} + \frac{m+2}{m}\right)$$

$$= 3$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = 3$$

(ii) When $n = 2m$,

$$S_n = S_{2m}$$

$$= S_{2m-1} + \left(-\frac{m+3}{m+1}\right) = 3 - \frac{m+3}{m+1}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{1 + \frac{3}{m}}{1 + \frac{1}{m}} \right) = 3 - 1 = 2$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series diverges.

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Determine whether each given infinite series converges or diverges.

If it converges, find the sum.

Ex. $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m$,

$$\begin{aligned} S_n &= S_{2m} \\ &= \left[\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^m \right] + \left[\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots + \left(\frac{1}{3}\right)^m \right] \\ &= \frac{\frac{1}{2} \left[1 - \left(\frac{1}{2}\right)^m \right]}{1 - \frac{1}{2}} + \frac{\frac{1}{3} \left[1 - \left(\frac{1}{3}\right)^m \right]}{1 - \frac{1}{3}} \\ &= \left[1 - \left(\frac{1}{2}\right)^m \right] + \frac{1}{2} \left[1 - \left(\frac{1}{3}\right)^m \right] \end{aligned}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \boxed{\frac{3}{2}}$$

(ii) When $n = 2m - 1$,

$$\begin{aligned} S_n &= S_{2m-1} \\ &= S_{2m} - \left(\frac{1}{3}\right)^m \end{aligned}$$

$$S_{2m-1} = S_{2m} - (\text{the } 2m^{\text{th}} \text{ term})$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \lim_{m \rightarrow \infty} \left[S_{2m} - \left(\frac{1}{3}\right)^m \right] = \boxed{\frac{3}{2}}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} S_{2m-1}$

Therefore, the series converges and the sum is $\boxed{\frac{3}{2}}$.

From (i),

$$\lim_{m \rightarrow \infty} S_{2m} =$$

N98b

$$(1) \quad 1 + 2 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{2}{9} + \frac{1}{8} - \frac{2}{27} + \dots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m$,

$$S_n = S_{2m}$$

$$= \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{m-1} \right] + \left[2 - \frac{2}{3} + 2\left(-\frac{1}{3}\right)^2 + \dots + 2\left(-\frac{1}{3}\right)^{m-1} \right]$$

$$= \frac{1 - \left(\frac{1}{2}\right)^m}{1 - \frac{1}{2}} + \frac{2 \left[1 - \left(-\frac{1}{3}\right)^m \right]}{1 - \left(-\frac{1}{3}\right)}$$

$$= 2 \left[1 - \left(\frac{1}{2}\right)^m \right] + \frac{3}{2} \left[1 - \left(-\frac{1}{3}\right)^m \right]$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \frac{7}{2}$$

(ii) When $n = 2m - 1$,

$$S_n = S_{2m-1}$$

$$= S_{2m} - 2 \left(-\frac{1}{3}\right)^{m-1}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \lim_{m \rightarrow \infty} \left[S_{2m} - 2 \left(-\frac{1}{3}\right)^{m-1} \right] = \frac{7}{2}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} S_{2m-1} = \frac{7}{2}$

Therefore, the series converges and the sum is $\frac{7}{2}$.

Infinite Series

Name _____

Date / /

Time : to :

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(mistakes) 0	—	—	1	2

1. Given the sequence $\{a_n\}$: $\frac{3}{1 \cdot 2 \cdot 2^1}, \frac{4}{2 \cdot 3 \cdot 2^2}, \frac{5}{3 \cdot 4 \cdot 2^3}, \frac{6}{4 \cdot 5 \cdot 2^4}, \dots$
 solve the following questions.

- (1) Find the general term of $\{a_n\}$.

[Sol] $a_n = \frac{n+2}{n(n+1) \cdot 2^n}$

- (2) Find the constants b and c which satisfy $2^n a_n = \frac{b}{n} + \frac{c}{n+1}$.

[Sol] From (1), $2^n a_n = \frac{n+2}{n(n+1)} \dots \textcircled{1}$

Since $2^n a_n = \frac{b}{n} + \frac{c}{n+1}$,

$2^n a_n = \frac{(b+c)n+b}{n(n+1)} \dots \textcircled{2}$

From $\textcircled{1}$ and $\textcircled{2}$, $n+2 = (b+c)n+b$
 $\therefore b=2, c=-1$

Since $n+2 = (b+c)n+b$,
 $\begin{cases} b+c=1 \\ b=2 \end{cases}$

- (3) Find the sum S_n of the first n terms of $\{a_n\}$ and its limit value $\lim_{n \rightarrow \infty} S_n$.

[Sol] From (2), $a_n = \left(\frac{2}{n} - \frac{1}{n+1} \right) \frac{1}{2^n} = \frac{1}{n \cdot 2^{n-1}} - \frac{1}{(n+1) \cdot 2^n}$

$\therefore S_n = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{12} \right) + \left(\frac{1}{12} - \frac{1}{32} \right) + \dots + \left[\frac{1}{n \cdot 2^{n-1}} - \frac{1}{(n+1) \cdot 2^n} \right]$
 $= 1 - \frac{1}{(n+1) \cdot 2^n}$

$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{(n+1) \cdot 2^n} \right] = 1$

N99b

2. Find the sum of the following infinite series.

$$1 + \frac{2}{\pi} + \frac{3}{\pi^2} + \frac{4}{\pi^3} + \dots + \frac{n}{\pi^{n-1}} + \dots$$

(Use the condition $\lim_{n \rightarrow \infty} nx^n = 0$ when $|x| < 1$.)

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n = 1 + \frac{2}{\pi} + \frac{3}{\pi^2} + \frac{4}{\pi^3} + \dots + \frac{n}{\pi^{n-1}} \quad \text{--- ①}$$

Multiplying both sides of ① by $\frac{1}{\pi}$,

$$\frac{1}{\pi} S_n = \frac{1}{\pi} + \frac{2}{\pi^2} + \frac{3}{\pi^3} + \dots + \frac{n-1}{\pi^{n-1}} + \frac{n}{\pi^n} \quad \text{--- ②}$$

From ① - ②,

$$\begin{aligned} \left(1 - \frac{1}{\pi}\right) S_n &= \left(1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \dots + \frac{1}{\pi^{n-1}}\right) - \frac{n}{\pi^n} \\ &= \frac{1 - \left(\frac{1}{\pi}\right)^n}{1 - \frac{1}{\pi}} - \frac{n}{\pi^n} \end{aligned}$$

$$\begin{aligned} \therefore S_n &= \frac{1 - \left(\frac{1}{\pi}\right)^n}{\left(1 - \frac{1}{\pi}\right)^2} - \frac{1}{1 - \frac{1}{\pi}} \cdot \frac{n}{\pi^n} \\ &= \left(\frac{\pi}{\pi-1}\right)^2 \left[1 - \left(\frac{1}{\pi}\right)^n\right] - \frac{\pi}{\pi-1} \cdot \frac{n}{\pi^n} \end{aligned}$$

Therefore, the sum of the infinite series is

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left\{ \left(\frac{\pi}{\pi-1}\right)^2 \left[1 - \left(\frac{1}{\pi}\right)^n\right] - \frac{\pi}{\pi-1} \cdot \frac{n}{\pi^n} \right\} \\ &= \left(\frac{\pi}{\pi-1}\right)^2 \end{aligned}$$

From the condition, $\lim_{n \rightarrow \infty} nx^n = 0$ (2)

Also, $\left|\frac{1}{\pi}\right| < 1$ from $\pi = 3.14$.

$$\therefore \lim_{n \rightarrow \infty} \frac{n}{\pi^n} = \lim_{n \rightarrow \infty} n \left(\frac{1}{\pi}\right)^n = 0$$

Infinite Series

Name _____

Date ____/____/____

Time ____:____ to ____:____

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1. Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

➡ N9

$$(1) \sum_{n=1}^{\infty} \frac{2}{\sqrt{n+1} + \sqrt{n+2}}$$

$$[\text{Sol}] \frac{2}{\sqrt{n+1} + \sqrt{n+2}} = \frac{2(\sqrt{n+1} - \sqrt{n+2})}{(\sqrt{n+1} + \sqrt{n+2})(\sqrt{n+1} - \sqrt{n+2})}$$

$$= -2(\sqrt{n+1} - \sqrt{n+2})$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = -2(\sqrt{2} - \sqrt{3}) - 2(\sqrt{3} - \sqrt{4}) - 2(\sqrt{4} - \sqrt{5}) - \dots - 2(\sqrt{n+1} - \sqrt{n+2})$$

$$= -2(\sqrt{2} - \sqrt{n+2})$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} [-2(\sqrt{2} - \sqrt{n+2})] = \infty$$

Thus, the series **diverges**.

$$(2) \frac{4}{1 \cdot 5} + \frac{4}{5 \cdot 9} + \frac{4}{9 \cdot 13} + \dots + \frac{4}{(4n-3)(4n+1)} + \dots$$

➡ N

$$[\text{Sol}] \text{ Let } \frac{4}{(4n-3)(4n+1)} = \frac{a}{4n-3} - \frac{b}{4n+1}$$

$$4 = a(4n+1) - b(4n-3)$$

$$= 4(a-b)n + (a+3b)$$

$$\therefore a=1, b=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \dots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$$

$$= 1 - \frac{1}{4n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{4n+1}\right) = 1$$

Thus, the series **converges** and the sum is 1.

2. Find the sum of the following infinite series.

$$\sum_{n=1}^{\infty} \frac{5^n - 2^n}{10^n}$$

[Sol] $\sum_{n=1}^{\infty} \frac{5^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

$$\sum_{n=1}^{\infty} \frac{2^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \frac{\frac{1}{5}}{1 - \frac{1}{5}} = \frac{1}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{5^n - 2^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = 1 - \frac{1}{4} = \frac{3}{4}$$

3. Determine whether the following infinite series converges or diverges. If it converges, find the sum.

$$\frac{2}{3} - \frac{4}{5} + \frac{4}{5} - \frac{6}{7} + \frac{6}{7} - \frac{2m}{2m+1} + \frac{2m}{2m+1} - \frac{2m+2}{2m+3} + \dots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$\begin{aligned} S_n &= S_{2m-1} \\ &= \frac{2}{3} + \left(-\frac{4}{5} + \frac{4}{5}\right) + \left(-\frac{6}{7} + \frac{6}{7}\right) + \dots + \left(-\frac{2m}{2m+1} + \frac{2m}{2m+1}\right) \\ &= \frac{2}{3} \end{aligned}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \frac{2}{3}$$

(ii) When $n = 2m$,

$$\begin{aligned} S_n &= S_{2m} \\ &= S_{2m-1} + \left(-\frac{2m+2}{2m+3}\right) \\ \therefore \lim_{m \rightarrow \infty} S_{2m} &= \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{2 + \frac{2}{m}}{2 + \frac{3}{m}} \right) = -\frac{1}{3} \end{aligned}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series diverges.

Limits of Functions 1

Name _____

Date ____/____/____

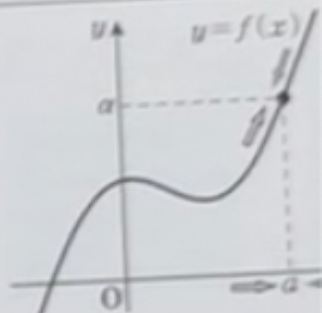
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Given the function $f(x)$, if $f(x)$ approaches a constant value α as x approaches a , then $f(x)$ is said to **converge** to α , where it is expressed as:

$$\lim_{x \rightarrow a} f(x) = \alpha \quad \text{or} \quad f(x) \rightarrow \alpha \text{ as } x \rightarrow a.$$

The value α is called the **limit** or the **limit value** of the function $f(x)$ as $x \rightarrow a$. As with limits of sequences, the following are true for limits of functions.



Properties of Limits of Functions

If $\lim_{x \rightarrow a} f(x) = \alpha$ and $\lim_{x \rightarrow a} g(x) = \beta$, then

$$\lim_{x \rightarrow a} k f(x) = k \alpha \quad (k \text{ is a constant})$$

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \alpha + \beta, \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \alpha - \beta$$

$$\lim_{x \rightarrow a} f(x) g(x) = \alpha \beta$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta} \quad (\beta \neq 0)$$

Find the following limits.

Ex. $\lim_{x \rightarrow 3} \sqrt{2x-1} = \sqrt{2 \cdot 3 - 1} = \sqrt{5}$

(1) $\lim_{x \rightarrow 2} \sqrt{3x+5} = \sqrt{3 \cdot 2 + 5} = \sqrt{11}$

(2) $\lim_{x \rightarrow 1} 3^x = 3^1 = 3$

(3) $\lim_{x \rightarrow -1} \frac{x+2}{(x+4)(x-3)} = \frac{-1+2}{(-1+4)(-1-3)} = -\frac{1}{12}$

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \rightarrow 3} (x+3) = 6$$

$$\lim_{x \rightarrow -5} \frac{x^2 + 2x - 15}{x + 5} = \lim_{x \rightarrow -5} \frac{(x+5)(x-3)}{x+5} = \lim_{x \rightarrow -5} (x-3) = -8$$

$$\lim_{x \rightarrow 0} \frac{x^3 + x}{3x - x^2} = \lim_{x \rightarrow 0} \frac{x(x^2 + 1)}{x(3 - x)} = \lim_{x \rightarrow 0} \frac{x^2 + 1}{3 - x} = \frac{1}{3}$$

$$(6) \lim_{x \rightarrow -2} \frac{x^3 + 8}{x^2 - x - 6} = \lim_{x \rightarrow -2} \frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x-3)} = \lim_{x \rightarrow -2} \frac{x^2 - 2x + 4}{x-3} = \frac{12}{-5}$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$(7) \lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^3 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x-3)}{(x-1)(x^2 + x + 1)} = \lim_{x \rightarrow 1} \frac{x-3}{x^2 + x + 1} = -\frac{2}{3}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \left(1 - \frac{1}{x+1} \right) = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{x}{x+1} = \lim_{x \rightarrow 0} \frac{1}{x+1} = 1$$

$$1 - \frac{1}{x+1} = \frac{(x+1) - 1}{x+1} = \frac{x}{x+1}$$

Limits of Functions 1

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Find the following limits.

Ex.

$$\begin{aligned}
 \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} &= \lim_{x \rightarrow 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)} \\
 &= \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)} \\
 &= \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} \\
 &= \frac{1}{4}
 \end{aligned}$$

Multiplying the numerator and the denominator by $\sqrt{x+1}+2$

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 4} \frac{\sqrt{x+5}-3}{x-4} &= \lim_{x \rightarrow 4} \frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(x-4)(\sqrt{x+5}+3)} \\
 &= \lim_{x \rightarrow 4} \frac{x-4}{(x-4)(\sqrt{x+5}+3)} \\
 &= \lim_{x \rightarrow 4} \frac{1}{\sqrt{x+5}+3} \\
 &= \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})} \\
 &= \lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} \\
 &= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} &= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x}+1)}{x-1} \\
 &= \lim_{x \rightarrow 1} (\sqrt{x}+1) \\
 &= 2
 \end{aligned}$$

Multiplying the numerator and the denominator by $\sqrt{x}+1$

$$\begin{aligned}
 (4) \quad & \lim_{x \rightarrow 4} \frac{x}{\sqrt{x+5} - \sqrt{5}} \\
 &= \lim_{x \rightarrow 4} \frac{x(\sqrt{x+5} + \sqrt{5})}{(\sqrt{x+5} - \sqrt{5})(\sqrt{x+5} + \sqrt{5})} \\
 &= \lim_{x \rightarrow 4} \frac{x(\sqrt{x+5} + \sqrt{5})}{x}
 \end{aligned}$$

$$= \lim_{x \rightarrow 4} (\sqrt{x+5} + \sqrt{5})$$

$$= 2\sqrt{5}$$

$$(5) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2}-1}{\sqrt{1+x}-\sqrt{1-x}}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x}+\sqrt{1-x})}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x+x^2}+1)(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x+x^2}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{x(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x+x^2}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2(\sqrt{1+x+x^2}+1)}$$

$$= \frac{1}{2}$$

Multiplying the numerator and the denominator by $\sqrt{1+x}+\sqrt{1-x}$

Multiplying the numerator and the denominator by $\sqrt{1+x+x^2}+1$

Alternative Solution

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2}-1}{\sqrt{1+x}-\sqrt{1-x}} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x+x^2}+1)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x+x^2}+1)}$$

$$\lim_{x \rightarrow 0} \frac{x(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x+x^2}+1)} = \lim_{x \rightarrow 0} \frac{(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2(\sqrt{1+x+x^2}+1)} = \frac{1}{2}$$

Summary

If the limit becomes an indeterminate form $\frac{0}{0}$, the expression has to be

Limits of Functions 1

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Date ____/____/____

Time ____:____ to ____:____

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Given $f(x) = \frac{1}{x^2}$, solve the following questions. Circle the correct term in the brackets () below.

[1] $f\left(\frac{1}{2}\right) = 4$

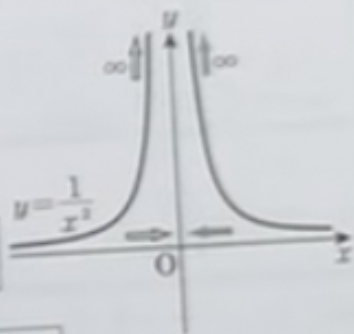
[4] $f\left(-\frac{1}{2}\right) = 4$

[2] $f\left(\frac{1}{10}\right) = 100$

[5] $f\left(-\frac{1}{10}\right) = 100$

[3] $f\left(\frac{1}{100}\right) = 10000$

[6] $f\left(-\frac{1}{100}\right) = 10000$



[7] From the answers above, when the value of x approaches 0, the value of $f(x) = \frac{1}{x^2}$ becomes (larger, smaller).

$$\therefore \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

SWTCS: 100, 10000, 4, 100, 10000, larger, ∞

1. Given $f(x) = -\frac{1}{x^2}$, solve the following questions.

[1] $f\left(\frac{1}{2}\right) = -4$

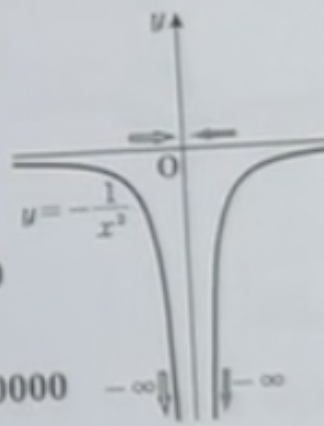
[4] $f\left(-\frac{1}{2}\right) = -4$

[2] $f\left(\frac{1}{10}\right) = -100$

[5] $f\left(-\frac{1}{10}\right) = -100$

[3] $f\left(\frac{1}{100}\right) = -10000$

[6] $f\left(-\frac{1}{100}\right) = -10000$



2. Using the answers from question 1, find the following limit.

$$\lim_{x \rightarrow 0} \left(-\frac{1}{x^2}\right) = -\infty$$

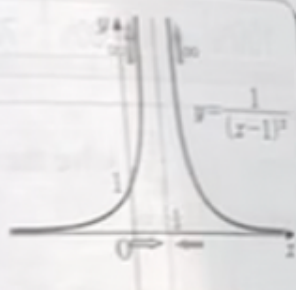
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and the following limits.

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$$

[Sol] As $x \rightarrow 1$, $(x-1)^2 \rightarrow 0$

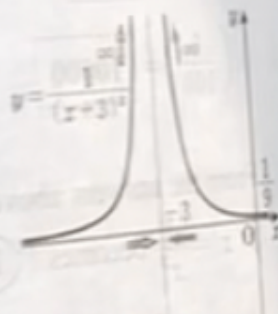
$$\therefore \lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$$



$$\lim_{x \rightarrow -3} \frac{1}{(x+3)^2}$$

[Sol] As $x \rightarrow -3$, $(x+3)^2 \rightarrow 0$

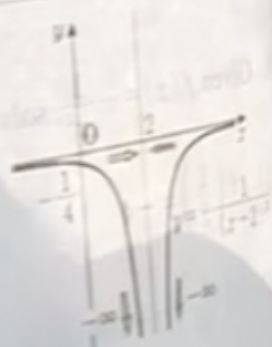
$$\therefore \lim_{x \rightarrow -3} \frac{1}{(x+3)^2} = \infty$$



$$(2) \lim_{x \rightarrow 2} \left[-\frac{1}{(x-2)^2} \right]$$

[Sol] As $x \rightarrow 2$, $(x-2)^2 \rightarrow 0$

$$\therefore \lim_{x \rightarrow 2} \left[-\frac{1}{(x-2)^2} \right] = -\infty$$

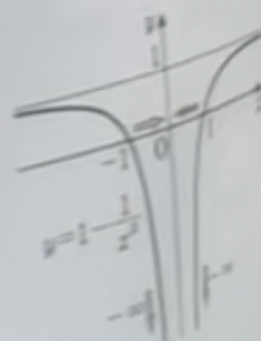


$$(3) \lim_{x \rightarrow 0} \left(1 - \frac{1}{x^2} \right)$$

[Sol] As $x \rightarrow 0$, $x^2 \rightarrow 0$

$$\therefore \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

$$\therefore \lim_{x \rightarrow 0} \left(1 - \frac{1}{x^2} \right) = -\infty$$



Limits of Functions 1

Name _____

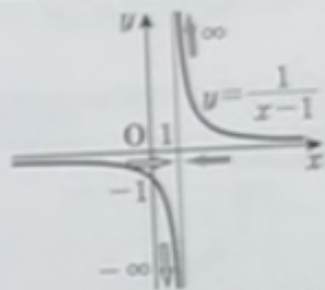
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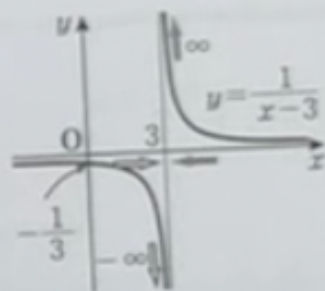
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1. Determine the following limits for each different case of $\dot{x}$.**Ex.**

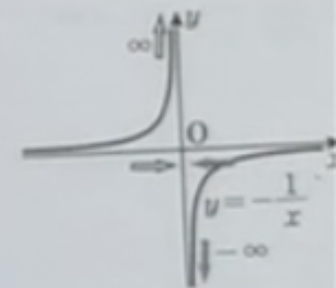
$$\lim_{x \rightarrow 1} \frac{1}{x-1}$$

[Sol] (i) When $x > 1$, as $x \rightarrow 1$, $\frac{1}{x-1} \rightarrow \infty$ (ii) When $x < 1$, as $x \rightarrow 1$, $\frac{1}{x-1} \rightarrow -\infty$ 

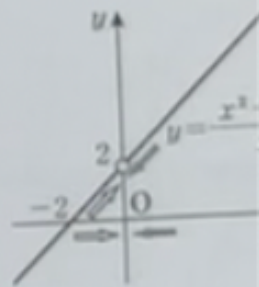
$$(1) \lim_{x \rightarrow 3} \frac{1}{x-3}$$

[Sol] (i) When $x > 3$, as $x \rightarrow 3$, $\frac{1}{x-3} \rightarrow \infty$ (ii) When $x < 3$, as $x \rightarrow 3$, $\frac{1}{x-3} \rightarrow -\infty$ 

$$(2) \lim_{x \rightarrow 0} \left(-\frac{1}{x}\right)$$

[Sol] (i) When $x > 0$, as $x \rightarrow 0$, $-\frac{1}{x} \rightarrow -\infty$ (ii) When $x < 0$, as $x \rightarrow 0$, $-\frac{1}{x} \rightarrow \infty$ 

$$(3) \lim_{x \rightarrow 0} \frac{x^2+2x}{x}$$

[Sol] When $x \neq 0$, $\frac{x^2+2x}{x} = x+2$ (i) When $x > 0$, as $x \rightarrow 0$, $\frac{x^2+2x}{x} = x+2 \rightarrow 2$ (ii) When $x < 0$, as $x \rightarrow 0$, $\frac{x^2+2x}{x} = x+2 \rightarrow 2$ 

When $x=0$, $\frac{x^2+2x}{x}$ is not defined.

For the function $f(x)$, the limit when x approaches a from the right is called the **right-handed limit** and is expressed as $\lim_{x \rightarrow a^+} f(x)$.

The limit when x approaches a from the left is called the **left-handed limit** and is expressed as $\lim_{x \rightarrow a^-} f(x)$.



Find the following limits.

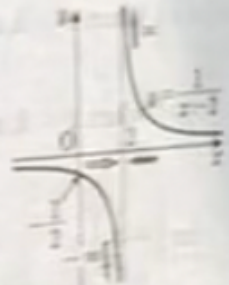
$$\lim_{x \rightarrow 4^+} \frac{1}{x-4} = \infty$$

$$\lim_{x \rightarrow 4^-} \frac{1}{x-4} = -\infty$$



$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$$



$$\lim_{x \rightarrow -1^+} \frac{1}{(x+1)^2} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{1}{(x+1)^2} = \infty$$



Limits of Functions 1

Name _____

Date ____/____/____

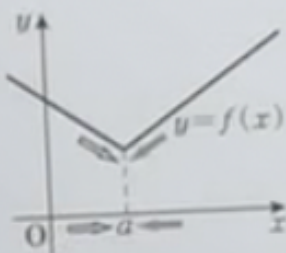
Time ____:____ to ____:____

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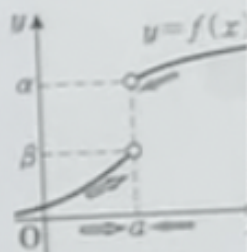
If $\lim_{x \rightarrow a^+} f(x)$ is equal to $\lim_{x \rightarrow a^-} f(x)$, then $\lim_{x \rightarrow a} f(x)$ exists.

Also, if $\lim_{x \rightarrow a^+} f(x)$ is not equal to $\lim_{x \rightarrow a^-} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$



$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$$



From the above, the following statements are true.

Existence of a Limit

If $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = \alpha$, then $\lim_{x \rightarrow a} f(x) = \alpha$ exists.

If $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

Determine if the following limits exist by finding the right-handed and left-hand limits. If they exist, find the limits.

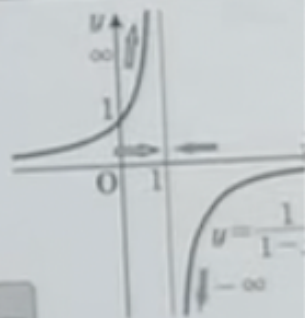
Ex.

$$\lim_{x \rightarrow 1} \frac{1}{1-x}$$

$$[\text{Sol}] \lim_{x \rightarrow 1^-} \frac{1}{1-x} = -\infty, \quad \lim_{x \rightarrow 1^+} \frac{1}{1-x} = \infty$$

Therefore, $\lim_{x \rightarrow 1} \frac{1}{1-x}$ does not exist. $\leftarrow$

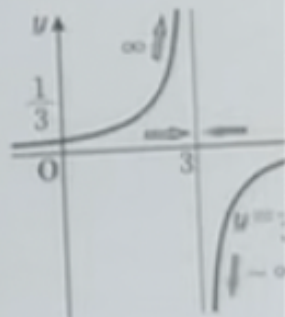
$$\lim_{x \rightarrow 1^-} \frac{1}{1-x} \neq \lim_{x \rightarrow 1^+} \frac{1}{1-x}$$



$$(1) \lim_{x \rightarrow 3} \frac{1}{3-x}$$

$$[\text{Sol}] \lim_{x \rightarrow 3^-} \frac{1}{3-x} = -\infty, \quad \lim_{x \rightarrow 3^+} \frac{1}{3-x} = \infty$$

Therefore, $\lim_{x \rightarrow 3} \frac{1}{3-x}$ does not exist.

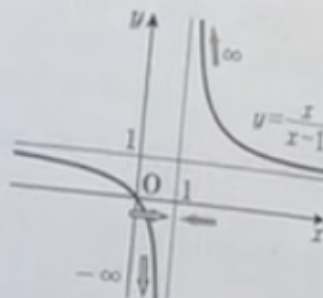


N105b

(2) $\lim_{x \rightarrow 1} \frac{x}{x-1}$

[Sol] $\lim_{x \rightarrow 1^+} \frac{x}{x-1} = \infty$, $\lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty$

Therefore, $\lim_{x \rightarrow 1} \frac{x}{x-1}$ does not exist.

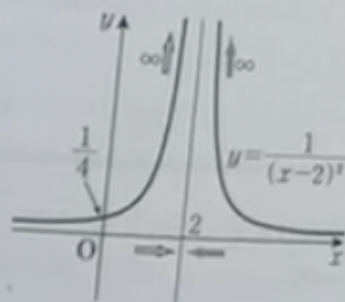


(3) $\lim_{x \rightarrow 2} \frac{1}{(x-2)^2}$

[Sol] $\lim_{x \rightarrow 2^+} \frac{1}{(x-2)^2} = \infty$, $\lim_{x \rightarrow 2^-} \frac{1}{(x-2)^2} = \infty$

Therefore, $\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty$

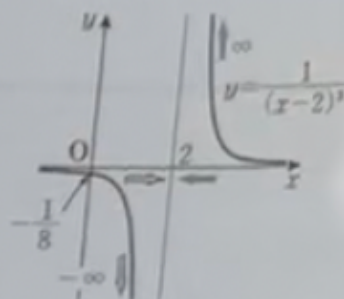
$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \lim_{x \rightarrow 2} \frac{1}{(x-2)^2}$



$\lim_{x \rightarrow 2} \frac{1}{(x-2)^3}$

$\lim_{x \rightarrow 2^+} \frac{1}{(x-2)^3} = \infty$, $\lim_{x \rightarrow 2^-} \frac{1}{(x-2)^3} = -\infty$

Therefore, $\lim_{x \rightarrow 2} \frac{1}{(x-2)^3}$ does not exist.



Limits of Functions 1

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Determine if the following limits exist. If they exist, find the limits.

Ex.

$$\lim_{x \rightarrow 0} \frac{x^2 + x}{|x|}$$

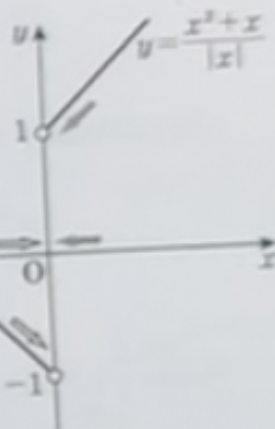
[Sol] (i) When $x > 0$, $\frac{x^2 + x}{|x|} = \frac{x(x+1)}{x} = x+1$

$$\therefore \lim_{x \rightarrow 0^+} \frac{x^2 + x}{|x|} = \lim_{x \rightarrow 0^+} (x+1) = 1$$

(ii) When $x < 0$, $\frac{x^2 + x}{|x|} = \frac{x(x+1)}{-x} = -x-1$

$$\therefore \lim_{x \rightarrow 0^-} \frac{x^2 + x}{|x|} = \lim_{x \rightarrow 0^-} (-x-1) = -1$$

From (i) and (ii), $\lim_{x \rightarrow 0} \frac{x^2 + x}{|x|}$ does not exist.



$$\lim_{x \rightarrow 0^+} \frac{x^2 + x}{|x|} \neq \lim_{x \rightarrow 0^-} \frac{x^2 + x}{|x|}$$

(1) $\lim_{x \rightarrow 0} \frac{x^2 - 2x}{|x|}$

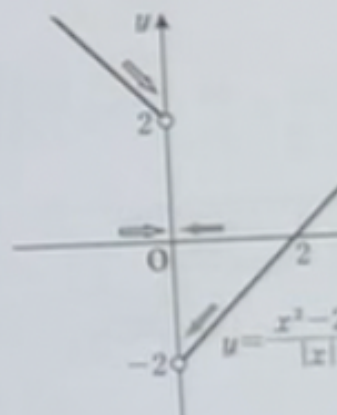
[Sol] (i) When $x > 0$, $\frac{x^2 - 2x}{|x|} = \frac{x(x-2)}{x} = x-2$

$$\therefore \lim_{x \rightarrow 0^+} \frac{x^2 - 2x}{|x|} = \lim_{x \rightarrow 0^+} (x-2) = -2$$

(ii) When $x < 0$, $\frac{x^2 - 2x}{|x|} = \frac{x(x-2)}{-x} = -x+2$

$$\therefore \lim_{x \rightarrow 0^-} \frac{x^2 - 2x}{|x|} = \lim_{x \rightarrow 0^-} (-x+2) = 2$$

From (i) and (ii), $\lim_{x \rightarrow 0} \frac{x^2 - 2x}{|x|}$ does not exist.



$$\frac{1}{-1}$$

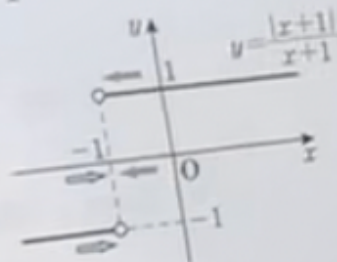
When $x+1 > 0$, i.e. $x > -1$, $\frac{|x+1|}{x+1} = \frac{x+1}{x+1} = 1$

$$\therefore \lim_{x \rightarrow -1^+} \frac{|x+1|}{x+1} = 1$$

When $x+1 < 0$, i.e. $x < -1$, $\frac{|x+1|}{x+1} = \frac{-(x+1)}{x+1} = -1$

$$\therefore \lim_{x \rightarrow -1^-} \frac{|x+1|}{x+1} = -1$$

From (i) and (ii), $\lim_{x \rightarrow -1} \frac{|x+1|}{x+1}$ does not exist.



$$\lim_{x \rightarrow 1} \frac{(x-1)^2}{|x^2-1|}$$

(i) When $x^2-1 > 0$, i.e. $x < -1$, $1 < x$,

$$\frac{(x-1)^2}{|x^2-1|} = \frac{(x-1)^2}{x^2-1} = \frac{(x-1)^2}{(x+1)(x-1)} = \frac{x-1}{x+1}$$

$$\therefore \lim_{x \rightarrow 1^-} \frac{(x-1)^2}{|x^2-1|} = \lim_{x \rightarrow 1^-} \frac{x-1}{x+1} = 0$$

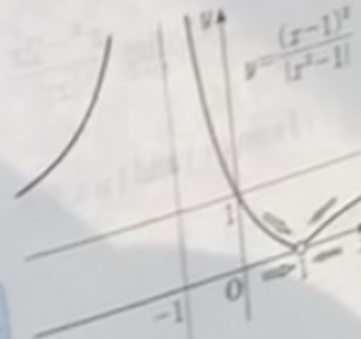
(ii) When $x^2-1 < 0$, i.e. $-1 < x < 1$,

$$\frac{(x-1)^2}{|x^2-1|} = \frac{(x-1)^2}{-(x^2-1)} = \frac{(x-1)^2}{-(x+1)(x-1)} = -\frac{x-1}{x+1}$$

$$\therefore \lim_{x \rightarrow 1^+} \frac{(x-1)^2}{|x^2-1|} = \lim_{x \rightarrow 1^+} \left(-\frac{x-1}{x+1} \right) = 0$$

From (i) and (ii), $\lim_{x \rightarrow 1} \frac{(x-1)^2}{|x^2-1|} = 0$

$$\lim_{x \rightarrow 1} \frac{(x-1)^2}{|x^2-1|} = \lim_{x \rightarrow 1} \frac{(x-1)^2}{|x^2-1|}$$



06b

$$\lim_{x \rightarrow -1} \frac{|x+1|}{x+1}$$

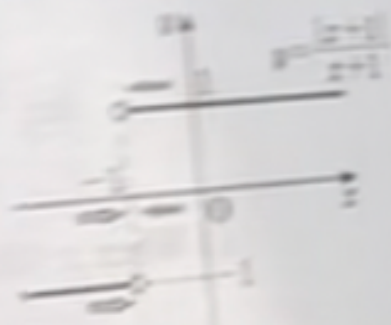
(i) When $x+1 > 0$, i.e. $x > -1$, $\frac{|x+1|}{x+1} = \frac{x+1}{x+1} = 1$

$$\therefore \lim_{x \rightarrow -1^+} \frac{|x+1|}{x+1} = 1$$

(ii) When $x+1 < 0$, i.e. $x < -1$, $\frac{|x+1|}{x+1} = \frac{-(x+1)}{x+1} = -1$

$$\therefore \lim_{x \rightarrow -1^-} \frac{|x+1|}{x+1} = -1$$

From (i) and (ii), $\lim_{x \rightarrow -1} \frac{|x+1|}{x+1}$ does not exist.



(3) $\lim_{x \rightarrow 1} \frac{(x-1)^2}{|x^2-1|}$

$x < 1$ i.e. $x < -1$ $1 < x$

Limits of Functions 1

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$[x]$ denotes the greatest integer less than or equal to a real number x .

This can be expressed as follows:

If n is an integer and $n \leq x < n+1$, then $[x] = n$

For example, $\left[\frac{7}{2}\right] = 3$, $[2] = 2$, $[0.99] = 0$, $\left[-\frac{1}{10}\right] = -1$.

Determine if the following limits exist. If they exist, find the limits.

Ex. $\lim_{x \rightarrow 2} [x]$

[Sol] (i) When $1 \leq x < 2$, $[x] = 1$

$$\therefore \lim_{x \rightarrow 2^-} [x] = 1$$

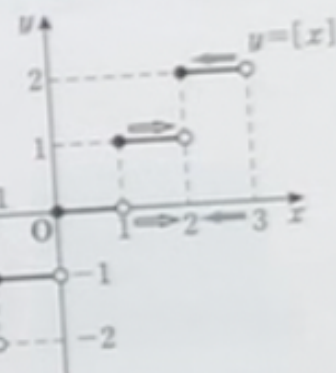
Considering $x \rightarrow 2^-$

(ii) When $2 \leq x < 3$, $[x] = 2$

$$\therefore \lim_{x \rightarrow 2^+} [x] = 2$$

Considering $x \rightarrow 2^+$

From (i) and (ii), $\lim_{x \rightarrow 2} [x]$ does not exist.



$$\lim_{x \rightarrow 2^-} [x] \neq \lim_{x \rightarrow 2^+} [x]$$

(1) $\lim_{x \rightarrow -1} [x]$

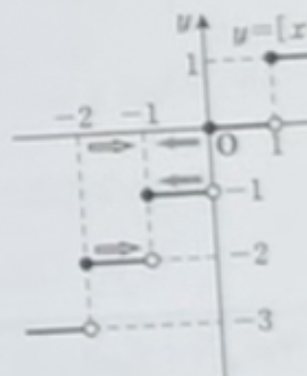
[Sol] (i) When $-2 \leq x < -1$, $[x] = -2$

$$\therefore \lim_{x \rightarrow -1^-} [x] = -2$$

(ii) When $-1 \leq x < 0$, $[x] = -1$

$$\therefore \lim_{x \rightarrow -1^+} [x] = -1$$

From (i) and (ii), $\lim_{x \rightarrow -1} [x]$ does not exist.



N107b

(2) $\lim_{x \rightarrow 2} \frac{[x]}{x}$

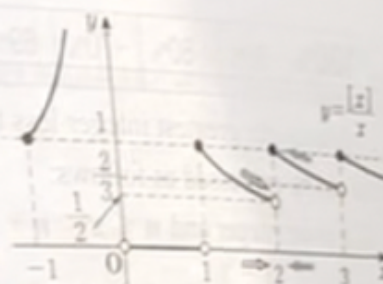
[Sol] (i) When $1 \leq x < 2$, $[x] = 1$

$$\therefore \lim_{x \rightarrow 2^-} \frac{[x]}{x} = \frac{1}{2}$$

(ii) When $2 \leq x < 3$, $[x] = 2$

$$\therefore \lim_{x \rightarrow 2^+} \frac{[x]}{x} = \frac{2}{2} = 1$$

From (i) and (ii), $\lim_{x \rightarrow 2} \frac{[x]}{x}$ does not exist.



(3) $\lim_{x \rightarrow 1} ([2x] - [x])$

[Sol] (i) When $\frac{1}{2} \leq x < 1$, $[x] = 0$

Since $1 \leq 2x < 2$, $[2x] = 1$

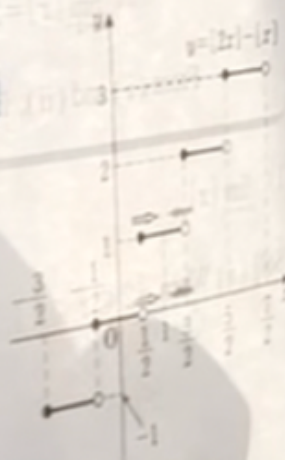
$$\therefore \lim_{x \rightarrow 1^-} ([2x] - [x]) = 1 - 0 = 1$$

(ii) When $1 \leq x < \frac{3}{2}$, $[x] = 1$

Since $2 \leq 2x < 3$, $[2x] = 2$

$$\therefore \lim_{x \rightarrow 1^+} ([2x] - [x]) = 2 - 1 = 1$$

From (i) and (ii), $\lim_{x \rightarrow 1} ([2x] - [x]) = 1$



✱ As for the interval $0 \leq x < 1$,
when $0 \leq x < \frac{1}{2}$, $[2x] = 0$

when $\frac{1}{2} \leq x < 1$, $[2x] = 1$

Therefore, $[2x]$ is not defined.

Thus, it is necessary to consider an interval where $[2x]$ is defined.

Limits of Functions 1

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Find the constants a and b which satisfy each given equation.

Ex. $\lim_{x \rightarrow 4} \frac{ax+b}{\sqrt{x}-2} = 8 \dots \textcircled{1}$

[Sol] Since $\lim_{x \rightarrow 4} (\sqrt{x} - 2) = 0$,

$$\lim_{x \rightarrow 4} (ax + b) = 0, \text{ i.e. } 4a + b = 0$$

$$\therefore b = -4a \cdots \textcircled{2}$$

Then,

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{ax+b}{\sqrt{x}-2} &= \lim_{x \rightarrow 4} \frac{a(x-4)}{\sqrt{x}-2} \\ &= \lim_{x \rightarrow 4} \frac{a(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 4} \frac{a(x-4)(\sqrt{x}+2)}{x-4} \\ &= \lim_{x \rightarrow 4} a(\sqrt{x}+2) \\ &= 4a \cdots \textcircled{3}\end{aligned}$$

Multiplying the numerator and the denominator by $\sqrt{x}+2$

From ① and ③, $4a = 8$; therefore, $a = 2 \dots \textcircled{4}$

From ② and ④, $b = \boxed{-8}$

$$\therefore a = \boxed{2}, b = \boxed{-8}$$

$$\frac{\sqrt{x-1}+b}{x-2}=3 \dots \textcircled{1}$$

$$\text{e } \lim_{x \rightarrow 2} (x-2)=0,$$

$$a\sqrt{x-1}+b=0, \text{ i.e. } a+b=0$$

$$b=-a \dots \textcircled{2}$$

n,

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{a\sqrt{x-1}+b}{x-2} &= \lim_{x \rightarrow 2} \frac{a(\sqrt{x-1}-1)}{x-2} \\ &= \lim_{x \rightarrow 2} \frac{a(\sqrt{x-1}-1)(\sqrt{x-1}+1)}{(x-2)(\sqrt{x-1}+1)} \\ &= \lim_{x \rightarrow 2} \frac{a(x-2)}{(x-2)(\sqrt{x-1}+1)} \\ &= \lim_{x \rightarrow 2} \frac{a}{\sqrt{x-1}+1} \\ &= \frac{a}{2} \dots \textcircled{3} \end{aligned}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{3}, \frac{a}{2}=3; \text{ therefore, } a=6 \dots \textcircled{4}$$

$$\text{From } \textcircled{2} \text{ and } \textcircled{4}, b=-6$$

$$\therefore a=6, b=-6$$

Summary

Let the functions $f(x)$, $g(x)$ and constant a ,

$$\text{e } \lim_{x \rightarrow a} \frac{f(x)}{g(x)}=a \text{ and also } \lim_{x \rightarrow a} g(x)=0,$$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \cdot g(x) \right] = a \cdot 0 = 0$$

$$\text{Therefore, if } \lim_{x \rightarrow a} \frac{f(x)}{g(x)}=a \text{ and } \lim_{x \rightarrow a} g(x)=0, \text{ then } \lim_{x \rightarrow a} f(x)=0$$

Limits of Functions 1

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100%	~90%	~80%	~70%	69%~
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1. Given $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - (1+ax)}{x^2} = b$ ($a \neq 0$), find the values of a and b .

[Sol] $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - (1+ax)}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{[\sqrt{1+x+x^2} - (1+ax)][\sqrt{1+x+x^2} + (1+ax)]}{x^2[\sqrt{1+x+x^2} + (1+ax)]}$$

Multiplying the numerator
and the denominator by
 $\sqrt{1+x+x^2} + (1+ax)$

$$= \lim_{x \rightarrow 0} \frac{(1-a^2)x^2 + (1-2a)x}{x^2(\sqrt{1+x+x^2} + 1+ax)}$$

$$= \lim_{x \rightarrow 0} \frac{(1-a^2)x + 1-2a}{x(\sqrt{1+x+x^2} + 1+ax)}$$

Since $\lim_{x \rightarrow 0} x(\sqrt{1+x+x^2} + 1+ax) = 0$,

$$\lim_{x \rightarrow 0} [(1-a^2)x + 1-2a] = 0, \text{ i.e. } 1-2a=0$$

$$\therefore a = \frac{1}{2}$$

Then,

$$\lim_{x \rightarrow 0} \frac{(1-a^2)x + 1-2a}{x(\sqrt{1+x+x^2} + 1+ax)} = \lim_{x \rightarrow 0} \frac{\left[1 - \left(\frac{1}{2}\right)^2\right]x + 1 - 2 \cdot \frac{1}{2}}{x\left(\sqrt{1+x+x^2} + 1 + \frac{1}{2}x\right)}$$

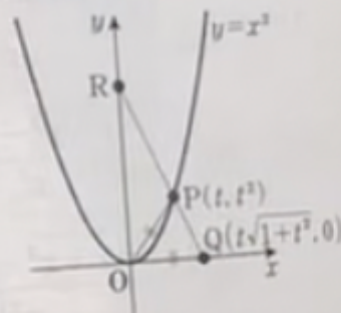
$$= \lim_{x \rightarrow 0} \frac{3}{4\left(\sqrt{1+x+x^2} + 1 + \frac{1}{2}x\right)}$$

$$= \frac{3}{8}$$

$$\therefore b = \frac{3}{8}$$

$$\therefore a = \frac{1}{2}, b = \frac{3}{8}$$

When point P on curve $y=x^2$ and point Q on the positive x -axis move while satisfying $OP=OQ$, let R be the point where line PQ intersects with the y -axis. Given that point P is in the 1st Quadrant and approaches origin O, find the coordinates of the point which point R approaches.



[Sol] Let (t, t^2) be the coordinates of point P. ($t > 0$)

$$OP = \sqrt{t^2 + (t^2)^2} = \sqrt{t^2 + t^4} = t\sqrt{1+t^2}$$

Since $OP=OQ$, the coordinates of point Q are $(t\sqrt{1+t^2}, 0)$.

Therefore, the equation of line PQ is

$$y - t^2 = \frac{0 - t^2}{t\sqrt{1+t^2} - t}(x - t)$$

Equation of a Line II (M11)

Since point R lies on the y -axis, the x -coordinate is 0.

When $x=0$,

$$y = t^2 + \frac{t^2}{\sqrt{1+t^2} - 1} = \frac{t^2\sqrt{1+t^2}}{\sqrt{1+t^2} - 1}$$

Thus, the coordinates of point R are $\left(0, \frac{t^2\sqrt{1+t^2}}{\sqrt{1+t^2} - 1}\right)$.

Then,

$$\lim_{t \rightarrow 0^+} \frac{t^2\sqrt{1+t^2}}{\sqrt{1+t^2} - 1} = \lim_{t \rightarrow 0^+} \frac{t^2\sqrt{1+t^2}(\sqrt{1+t^2} + 1)}{(\sqrt{1+t^2} - 1)(\sqrt{1+t^2} + 1)}$$

$$= \lim_{t \rightarrow 0^+} \frac{t^2\sqrt{1+t^2}(\sqrt{1+t^2} + 1)}{t^2}$$

$$= \lim_{t \rightarrow 0^+} \sqrt{1+t^2}(\sqrt{1+t^2} + 1)$$

$$= 2$$

When point P in the 1st Quadrant approaches the origin, $t \rightarrow 0^+$

the coordinates of the point which point R approaches are $(0, 2)$.

Limits of Functions 1

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1. Find the following limits.

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 1} \frac{2x^2 + x - 3}{x^2 + 2x - 3} &= \lim_{x \rightarrow 1} \frac{(2x+3)(x-1)}{(x+3)(x-1)} \\
 &= \lim_{x \rightarrow 1} \frac{2x+3}{x+3} \\
 &= \frac{5}{4}
 \end{aligned}$$

➡ N101

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow -3} \frac{\sqrt{x+7}-2}{x+3} &= \lim_{x \rightarrow -3} \frac{(\sqrt{x+7}-2)(\sqrt{x+7}+2)}{(x+3)(\sqrt{x+7}+2)} \\
 &= \lim_{x \rightarrow -3} \frac{x+3}{(x+3)(\sqrt{x+7}+2)} \\
 &= \lim_{x \rightarrow -3} \frac{1}{\sqrt{x+7}+2} \\
 &= \frac{1}{4}
 \end{aligned}$$

➡ N101

2. Determine if the following limit exists. If it exists, find the limit. ➡ N110

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

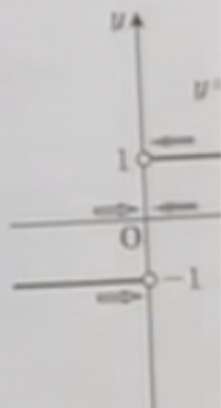
[Sol] (i) When $x > 0$, $\frac{|x|}{x} = \frac{x}{x} = 1$

$$\therefore \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$$

(ii) When $x < 0$, $\frac{|x|}{x} = \frac{-x}{x} = -1$

$$\therefore \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$$

From (i) and (ii), $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.



3. Find the constants a and b which satisfy the following equation. $\Rightarrow$ N108

$$\lim_{x \rightarrow 1} \frac{a\sqrt{x+1}-b}{x-1} = \sqrt{2} \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow 1} (x-1) = 0$,

$$\lim_{x \rightarrow 1} (a\sqrt{x+1}-b) = 0, \text{ i.e. } \sqrt{2}a - b = 0$$

$$\therefore b = \sqrt{2}a \quad \dots \textcircled{2}$$

Then,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{a\sqrt{x+1}-b}{x-1} &= \lim_{x \rightarrow 1} \frac{a(\sqrt{x+1}-\sqrt{2})}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{a(\sqrt{x+1}-\sqrt{2})(\sqrt{x+1}+\sqrt{2})}{(x-1)(\sqrt{x+1}+\sqrt{2})} \\ &= \lim_{x \rightarrow 1} \frac{a(x-1)}{(x-1)(\sqrt{x+1}+\sqrt{2})} \\ &= \lim_{x \rightarrow 1} \frac{a}{\sqrt{x+1}+\sqrt{2}} \\ &= \frac{a}{2\sqrt{2}} \\ &= \frac{\sqrt{2}}{4}a \quad \dots \textcircled{3} \end{aligned}$$

From ① and ③, $\frac{\sqrt{2}}{4}a = \sqrt{2}$; therefore, $a = 4 \quad \dots \textcircled{4}$

From ② and ④, $b = 4\sqrt{2}$

$$\therefore a = 4, b = 4\sqrt{2}$$

Therefore,

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Find the following limits.

Ex.

$$\lim_{x \rightarrow \infty} \frac{x+1}{x^2-3} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{3}{x^2}} = 0$$

Dividing the numerator and the denominator by the term with the highest power in the denominator (x^2 in this case)

$$(1) \quad \lim_{x \rightarrow \infty} \frac{2x+1}{x^3+x-1} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} + \frac{1}{x^3}}{1 + \frac{1}{x^2} - \frac{1}{x^3}} = 0$$

$$(2) \quad \lim_{x \rightarrow \infty} \frac{2x^2-x-6}{3x^2-2x-8} = \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x} - \frac{6}{x^2}}{3 - \frac{2}{x} - \frac{8}{x^2}} = \frac{2}{3}$$

$$(3) \quad \lim_{x \rightarrow \infty} \frac{5-7x+x^3}{4+3x^2-5x^3} = \lim_{x \rightarrow \infty} \frac{\frac{5}{x^3} - \frac{7}{x^2} + 1}{\frac{4}{x^3} + \frac{3}{x} - 5} = -\frac{1}{5}$$

$$(4) \quad \lim_{x \rightarrow \infty} \frac{x^2+1}{x-1} = \lim_{x \rightarrow \infty} \frac{x + \frac{1}{x}}{1 - \frac{1}{x}} = \infty$$

$$(5) \quad \lim_{x \rightarrow \infty} \frac{x-4x^2}{3+x} = \lim_{x \rightarrow \infty} \frac{1-4x}{\frac{3}{x}+1} = -\infty$$

$$\begin{aligned} \lim_{x \rightarrow \infty} (1-4x) &= -\infty \\ \lim_{x \rightarrow \infty} \left(\frac{3}{x} + 1 \right) &= 1 \end{aligned}$$

$$(6) \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{1}{1 + \sqrt{1 + \frac{1}{x^2}}} = \frac{1}{2} \leftarrow$$

Dividing the denominator
 $x + \sqrt{x^2 + 1}$ by x .

$$\frac{x + \sqrt{x^2 + 1}}{x} = 1 + \sqrt{\frac{x^2 + 1}{x^2}}$$

$$= 1 + \sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}$$

$$(7) \lim_{x \rightarrow \infty} \frac{x + \sqrt{5x^2 + 4}}{-2x + 1} = \lim_{x \rightarrow \infty} \frac{1 + \sqrt{5 + \frac{4}{x^2}}}{-2 + \frac{1}{x}} = -\frac{1 + \sqrt{5}}{2}$$

$$(8) \lim_{x \rightarrow \infty} \frac{3x^2 + \sqrt{x^3 + 2}}{x^2 - x + 1} = \lim_{x \rightarrow \infty} \frac{3 + \sqrt{\frac{1}{x} + \frac{2}{x^3}}}{1 - \frac{1}{x} + \frac{1}{x^2}} = 3$$

$$(9) \lim_{x \rightarrow \infty} \frac{\sqrt{x+1} + \sqrt{3x-1}}{\sqrt{x-1}} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}}}{\sqrt{1 - \frac{1}{x}}} = 1 + \sqrt{3} \leftarrow$$

Dividing the numerator and
the denominator by $\sqrt{x}$

$$(10) \lim_{x \rightarrow \infty} (x^2 - 3x + 2) = \lim_{x \rightarrow \infty} x^2 \left(1 - \frac{3}{x} + \frac{2}{x^2} \right) = \infty \leftarrow$$

Factoring by the term
with the highest power

$$\lim_{x \rightarrow \infty} (4 + 3x - 2x^2 - x^3) = \lim_{x \rightarrow \infty} x^3 \left(\frac{4}{x^3} + \frac{3}{x^2} - \frac{2}{x} - 1 \right) = -\infty \leftarrow$$

$$\lim_{x \rightarrow \infty} x^3 = \infty$$

$$\lim_{x \rightarrow \infty} \left(\frac{4}{x^3} + \frac{3}{x^2} - \frac{2}{x} - 1 \right) = -1$$

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1. Find the following limits.

Ex.

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x) = \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 2x} - x)(\sqrt{x^2 + 2x} + x)}{\sqrt{x^2 + 2x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2 + 2x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1}$$

$$= 1$$

Considering

$$\sqrt{x^2 + 2x} - x = \frac{\sqrt{x^2 + 2x} - x}{1}$$

then multiplying the numerator and the denominator by $\sqrt{x^2 + 2x} + x$

$$(1) \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 - x} - x) = \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 - x} - x)(\sqrt{x^2 - x} + x)}{\sqrt{x^2 - x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{-x}{\sqrt{x^2 - x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{-1}{\sqrt{1 - \frac{1}{x}} + 1}$$

$$= -\frac{1}{2}$$

$$(2) \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 + 1}) = \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 2x} - \sqrt{x^2 + 1})(\sqrt{x^2 + 2x} + \sqrt{x^2 + 1})}{\sqrt{x^2 + 2x} + \sqrt{x^2 + 1}}$$

$$= \lim_{x \rightarrow \infty} \frac{2x - 1}{\sqrt{x^2 + 2x} + \sqrt{x^2 + 1}}$$

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x}}{\sqrt{1 + \frac{2}{x}} + \sqrt{1 + \frac{1}{x^2}}}$$

$$= 1$$

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$$\begin{aligned}
 (3) \quad & \lim_{x \rightarrow \infty} (2x+1 - \sqrt{4x^2+2x+1}) \\
 &= \lim_{x \rightarrow \infty} \frac{(2x+1 - \sqrt{4x^2+2x+1})(2x+1 + \sqrt{4x^2+2x+1})}{2x+1 + \sqrt{4x^2+2x+1}} \\
 &= \lim_{x \rightarrow \infty} \frac{2x}{2x+1 + \sqrt{4x^2+2x+1}} \\
 &= \lim_{x \rightarrow \infty} \frac{2}{2 + \frac{1}{x} + \sqrt{4 + \frac{2}{x} + \frac{1}{x^2}}} \\
 &= \frac{1}{2}
 \end{aligned}$$

Find the constant a which satisfies $\lim_{x \rightarrow \infty} (\sqrt{x^2+ax+5} - \sqrt{x^2-3}) = 4$.

$$\begin{aligned}
 (1) \quad & \lim_{x \rightarrow \infty} (\sqrt{x^2+ax+5} - \sqrt{x^2-3}) \\
 &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2+ax+5} - \sqrt{x^2-3})(\sqrt{x^2+ax+5} + \sqrt{x^2-3})}{\sqrt{x^2+ax+5} + \sqrt{x^2-3}} \\
 &= \lim_{x \rightarrow \infty} \frac{ax+8}{\sqrt{x^2+ax+5} + \sqrt{x^2-3}} \\
 &= \lim_{x \rightarrow \infty} \frac{a + \frac{8}{x}}{\sqrt{1 + \frac{a}{x} + \frac{5}{x^2}} + \sqrt{1 - \frac{3}{x^2}}} \\
 &= \frac{a}{2} \\
 \therefore \quad & \frac{a}{2} = 4 \\
 \therefore \quad & a = 8
 \end{aligned}$$

When an indeterminate form $\frac{\infty}{\infty}$ or $\infty - \infty$, the expression

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Find the following limits.

Ex. $\lim_{x \rightarrow -\infty} (4x^5 + x^2 + 4)$

[Sol] Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\lim_{x \rightarrow -\infty} (4x^5 + x^2 + 4) = \lim_{t \rightarrow \infty} (-4t^5 + t^2 + 4) \quad \leftarrow$$

$$\begin{aligned} x^5 &= (-t)^5 = -t^5 \\ x^2 &= (-t)^2 = t^2 \end{aligned}$$

$$= \lim_{t \rightarrow \infty} t^5 \left(-4 + \frac{1}{t^3} + \frac{4}{t^5} \right)$$

$$= -\infty$$

(1) $\lim_{x \rightarrow -\infty} (2x^4 - x^3 + 2x - 1)$

[Sol] Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\lim_{x \rightarrow -\infty} (2x^4 - x^3 + 2x - 1) = \lim_{t \rightarrow \infty} (2t^4 + t^3 - 2t - 1)$$

$$= \lim_{t \rightarrow \infty} t^4 \left(2 + \frac{1}{t} - \frac{2}{t^3} - \frac{1}{t^4} \right)$$

$$= \infty$$

(2) $\lim_{x \rightarrow -\infty} \frac{4x^2 - x + 2}{2x - 3}$

[Sol] Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\lim_{x \rightarrow -\infty} \frac{4x^2 - x + 2}{2x - 3} = \lim_{t \rightarrow \infty} \frac{4t^2 + t + 2}{-2t - 3}$$

$$= \lim_{t \rightarrow \infty} \frac{4t + 1 + \frac{2}{t}}{-2 - \frac{3}{t}}$$

$$= -\infty$$

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$$\frac{\sqrt{1+x^2}-1}{x}$$

Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{1+x^2}-1}{x} &= \lim_{t \rightarrow \infty} \frac{\sqrt{1+t^2}-1}{-t} \\ &= \lim_{t \rightarrow \infty} \left(-\sqrt{\frac{1}{t^2}+1} + \frac{1}{t} \right) \\ &= -1\end{aligned}$$

$$\lim_{x \rightarrow -\infty} (\sqrt{x^2+3x-1} - \sqrt{x^2-x+1})$$

Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\begin{aligned}\lim_{x \rightarrow -\infty} (\sqrt{x^2+3x-1} - \sqrt{x^2-x+1}) &= \lim_{t \rightarrow \infty} (\sqrt{t^2-3t-1} - \sqrt{t^2+t+1}) \\ &= \lim_{t \rightarrow \infty} \frac{(\sqrt{t^2-3t-1} - \sqrt{t^2+t+1})(\sqrt{t^2-3t-1} + \sqrt{t^2+t+1})}{\sqrt{t^2-3t-1} + \sqrt{t^2+t+1}} \\ &= \lim_{t \rightarrow \infty} \frac{-4t-2}{\sqrt{t^2-3t-1} + \sqrt{t^2+t+1}} \\ &= \lim_{t \rightarrow \infty} \frac{-4 - \frac{2}{t}}{\sqrt{1 - \frac{3}{t} - \frac{1}{t^2}} + \sqrt{1 + \frac{1}{t} + \frac{1}{t^2}}} \\ &= -2\end{aligned}$$

Considering

$$\frac{\sqrt{t^2-3t-1} - \sqrt{t^2+t+1}}{\sqrt{t^2-3t-1} + \sqrt{t^2+t+1}} = \frac{-4t-2}{1}$$

then multiplying the numerator and the denominator by $\sqrt{t^2-3t-1} + \sqrt{t^2+t+1}$

Note Summary

When $x \rightarrow -\infty$, it is easier to find the answer by letting $x = -t$ and considering the case when $t \rightarrow \infty$.
 $\sqrt{x^2} = -x$ when $x \leq 0$, and finding the correct answer becomes

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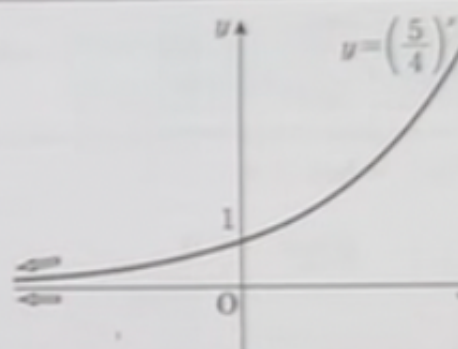
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Find the following limits.

Ex.

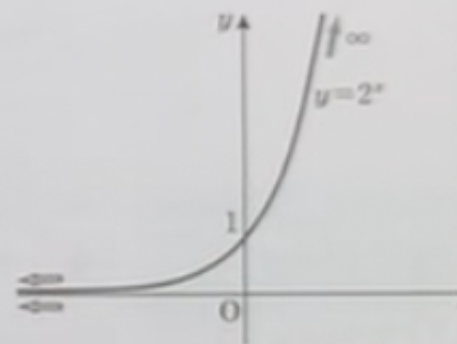
$$\lim_{x \rightarrow \infty} \left(\frac{5}{4}\right)^x = \infty$$

$$\lim_{x \rightarrow -\infty} \left(\frac{5}{4}\right)^x = 0$$



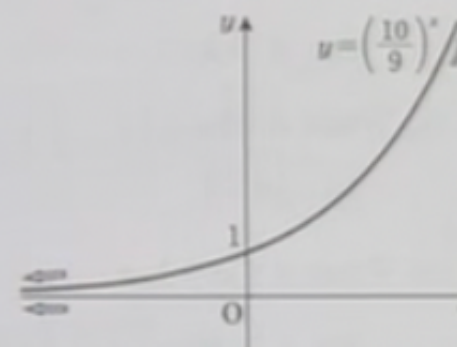
(1) $\lim_{x \rightarrow \infty} 2^x = \infty$

(2) $\lim_{x \rightarrow -\infty} 2^x = 0$



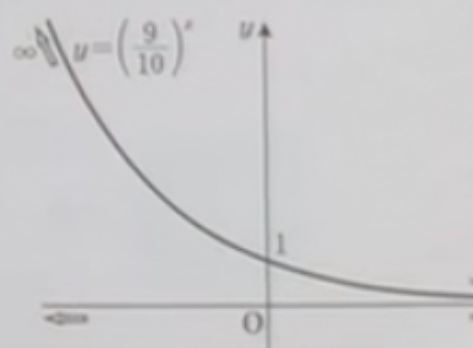
(3) $\lim_{x \rightarrow \infty} \left(\frac{10}{9}\right)^x = \infty$

(4) $\lim_{x \rightarrow -\infty} \left(\frac{10}{9}\right)^x = 0$



(5) $\lim_{x \rightarrow \infty} \left(\frac{9}{10}\right)^x = 0$

(6) $\lim_{x \rightarrow -\infty} \left(\frac{9}{10}\right)^x = \infty$



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(7) $\lim_{x \rightarrow \infty} a^x \quad (a > 0)$

[Sol] (i) When $a > 1$,

$$\lim_{x \rightarrow \infty} a^x = \boxed{\infty}$$

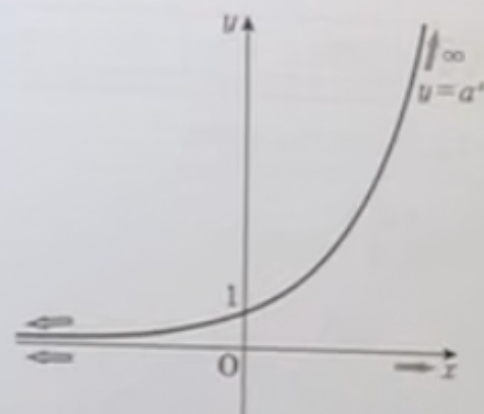
(ii) When $a = 1$,

$$\lim_{x \rightarrow \infty} a^x = \boxed{1}$$

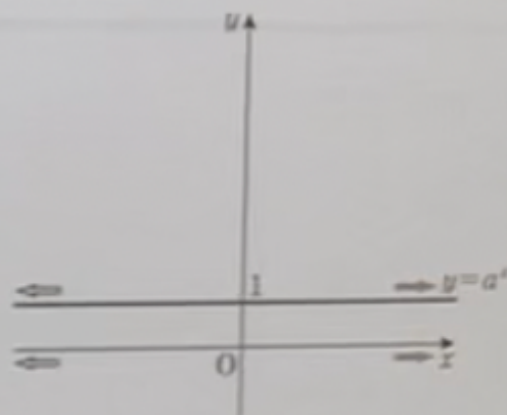
(iii) When $0 < a < 1$,

$$\lim_{x \rightarrow \infty} a^x = \boxed{0}$$

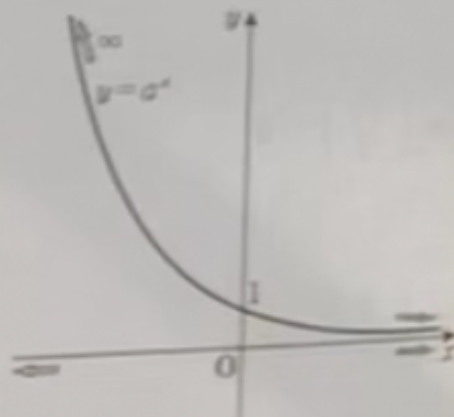
[When $a > 1$]



[When $a = 1$]



[When $0 < a < 1$]



(8) $\lim_{x \rightarrow -\infty} a^x \quad (a > 0)$

[Sol] (i) When $a > 1$,

$$\lim_{x \rightarrow -\infty} a^x = 0$$

(ii) When $a = 1$,

$$\lim_{x \rightarrow -\infty} a^x = 1$$

(iii) When $0 < a < 1$,

$$\lim_{x \rightarrow -\infty} a^x = \infty$$

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Find the following limits.

Ex.

$$\lim_{x \rightarrow \infty} (5^x - 3^x) = \lim_{x \rightarrow \infty} 5^x \left[1 - \left(\frac{3}{5} \right)^x \right] = \infty \quad \leftarrow$$

Factoring by the term with the largest absolute value of the b

$$(1) \quad \lim_{x \rightarrow \infty} (3^x - 2^x) = \lim_{x \rightarrow \infty} 3^x \left[1 - \left(\frac{2}{3} \right)^x \right] = \infty$$

$$(2) \quad \lim_{x \rightarrow \infty} (10^x - 3^x) = \lim_{x \rightarrow \infty} 10^x \left[1 - \left(\frac{3}{10} \right)^x \right] = \infty$$

$$(3) \quad \lim_{x \rightarrow \infty} (2^x - 5^x) = \lim_{x \rightarrow \infty} 5^x \left[\left(\frac{2}{5} \right)^x - 1 \right] = -\infty \quad \leftarrow$$

$$\begin{aligned} \lim_{x \rightarrow \infty} 5^x &= \infty \\ \lim_{x \rightarrow \infty} \left[\left(\frac{2}{5} \right)^x - 1 \right] &= -1 \end{aligned}$$

$$(4) \quad \lim_{x \rightarrow \infty} (3^x - 2^{2x}) = \lim_{x \rightarrow \infty} (3^x - 4^x) = \lim_{x \rightarrow \infty} 4^x \left[\left(\frac{3}{4} \right)^x - 1 \right] = -\infty$$

$$(5) \quad \lim_{x \rightarrow \infty} (2^{2x} - 3^x) = \lim_{x \rightarrow \infty} (4^x - 3^x) = \lim_{x \rightarrow \infty} 4^x \left[1 - \left(\frac{3}{4} \right)^x \right] = \infty$$

$$(6) \quad \lim_{x \rightarrow \infty} (5^x - 2^{-x}) = \lim_{x \rightarrow \infty} \left[5^x - \left(\frac{1}{2} \right)^x \right] = \infty \quad \leftarrow$$

$$\begin{aligned} \lim_{x \rightarrow \infty} 5^x &= \infty \\ \lim_{x \rightarrow \infty} \left(\frac{1}{2} \right)^x &= 0 \end{aligned}$$

$$\lim_{x \rightarrow \infty} \frac{2^{x-1}}{1+2^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^x + 1} = \frac{1}{2}$$

Dividing the numerator and the denominator by the term with the largest absolute value of the base in the denominator (2^x in this case)

$$\lim_{x \rightarrow \infty} \frac{4^x - 1}{4^x + 1} = \lim_{x \rightarrow \infty} \frac{1 - \left(\frac{1}{4}\right)^x}{1 + \left(\frac{1}{4}\right)^x} = 1$$

$$\lim_{x \rightarrow \infty} \frac{2^x - 3^{x+2}}{2^x + 3^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^x - 9}{\left(\frac{2}{3}\right)^x + 1} = -9$$

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^x - \left(\frac{2}{3}\right)^{-x}}{\left(\frac{2}{3}\right)^x + \left(\frac{2}{3}\right)^{-x}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^x - \left(\frac{3}{2}\right)^x}{\left(\frac{2}{3}\right)^x + \left(\frac{3}{2}\right)^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^{2x} - 1}{\left(\frac{2}{3}\right)^{2x} + 1} = -1$$

Since $\left(\frac{2}{3}\right)^{-x} = \left(\frac{3}{2}\right)^x$, dividing the numerator and the denominator by $\left(\frac{3}{2}\right)^x$

$$\lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} \quad (a > 0)$$

(i) When $a > 1$,

$$\lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} = \lim_{x \rightarrow \infty} \frac{1 + \left(\frac{1}{a}\right)^x}{1 + \left(\frac{1}{a}\right)^{x-1}} = 1 \quad \leftarrow \lim_{x \rightarrow \infty} \left(\frac{1}{a}\right)^x = 0$$

(ii) When $a = 1$,

$$\lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} = 1 \quad \leftarrow \lim_{x \rightarrow \infty} a^x = 1$$

(i) When $0 < a < 1$,

$$\lim_{x \rightarrow \infty} \frac{a^x + 1}{a^x + a} = \frac{1}{a} \quad \leftarrow \lim_{x \rightarrow \infty} a^x = 0$$

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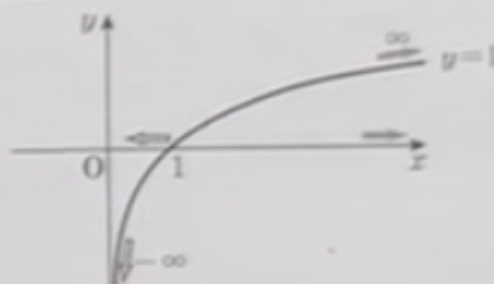
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Find the following limits.

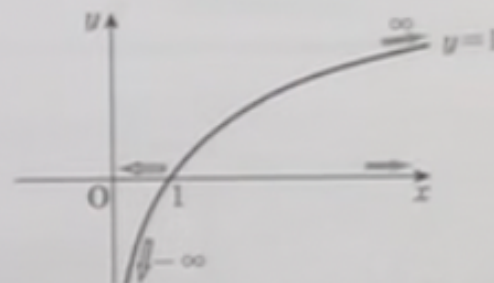
Ex.

$$\lim_{x \rightarrow \infty} \log_3 x = \infty$$

$$\lim_{x \rightarrow 0^+} \log_3 x = -\infty$$

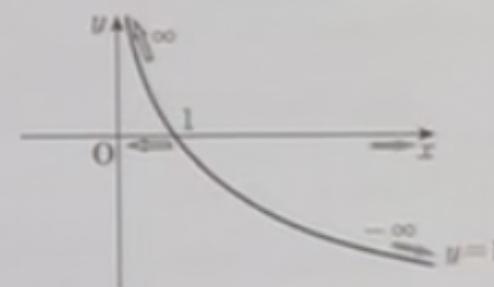


$$(1) \lim_{x \rightarrow \infty} \log_2 x = \infty$$



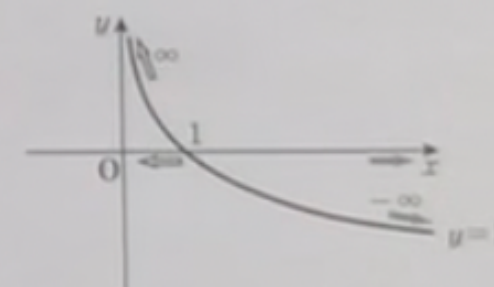
$$(2) \lim_{x \rightarrow 0^+} \log_2 x = -\infty$$

$$(3) \lim_{x \rightarrow \infty} \log_{\frac{1}{2}} x = -\infty$$



$$(4) \lim_{x \rightarrow 0^+} \log_{\frac{1}{2}} x = \infty$$

$$(5) \lim_{x \rightarrow \infty} \log_{\frac{1}{3}} x = -\infty$$



$$(6) \lim_{x \rightarrow 0^+} \log_{\frac{1}{3}} x = \infty$$

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7) $\lim_{x \rightarrow \infty} \log_a x \quad (a > 0, a \neq 1)$

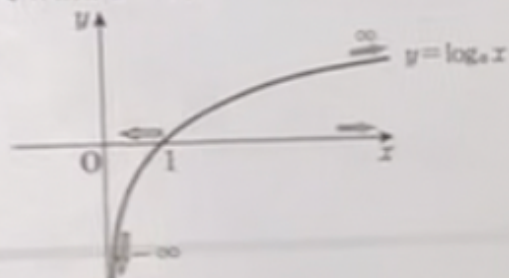
[Sol] (i) When $a > 1$,

$$\lim_{x \rightarrow \infty} \log_a x = \boxed{\infty}$$

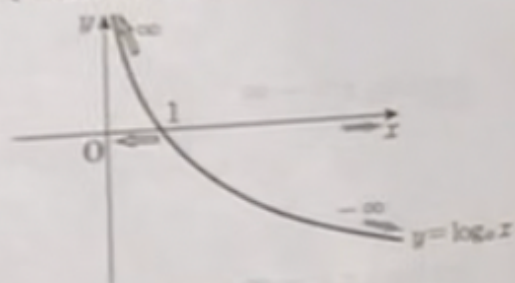
(ii) When $0 < a < 1$,

$$\lim_{x \rightarrow \infty} \log_a x = \boxed{-\infty}$$

[When $a > 1$]



[When $0 < a < 1$]



(8) $\lim_{x \rightarrow \frac{1}{a^+}} \log_a x \quad (a > 0, a \neq 1)$

[Sol] (i) When $a > 1$,

$$\lim_{x \rightarrow \frac{1}{a^+}} \log_a x = -\infty$$

(ii) When $0 < a < 1$,

$$\lim_{x \rightarrow \frac{1}{a^+}} \log_a x = \infty$$

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Find the following limits.

Ex.

$$\lim_{x \rightarrow \infty} [\log_2(4x+1) - \log_2(x-1)] = \lim_{x \rightarrow \infty} \log_2 \frac{4x+1}{x-1}$$

$$= \lim_{x \rightarrow \infty} \log_2 \frac{4 + \frac{1}{x}}{1 - \frac{1}{x}}$$

$$= \log_2 4$$

$$= 2$$

$$\log_a M - \log_a N = \log_a \frac{M}{N}$$

$$\log_a a^m = m$$

$$(1) \quad \lim_{x \rightarrow \infty} [\log_4(8x-2) - \log_4(2x+3)] = \lim_{x \rightarrow \infty} \log_4 \frac{8x-2}{2x+3}$$

$$= \lim_{x \rightarrow \infty} \log_4 \frac{8 - \frac{2}{x}}{2 + \frac{3}{x}}$$

$$= \log_4 4$$

$$= 1$$

$$\log_a a = 1$$

$$(2) \quad \lim_{x \rightarrow \infty} [\log_{\frac{1}{3}}(x+1) - \log_{\frac{1}{3}} x] = \lim_{x \rightarrow \infty} \log_{\frac{1}{3}} \frac{x+1}{x}$$

$$= \lim_{x \rightarrow \infty} \log_{\frac{1}{3}} \left(1 + \frac{1}{x} \right)$$

$$= \log_{\frac{1}{3}} 1$$

$$= 0$$

$$\log_a 1 = 0$$

$$\log_a a = 1, \quad \log_a 1 = 0, \quad \log_a M - \log_a N = \log_a \frac{M}{N}$$

1117b

$$1) \lim_{x \rightarrow \infty} (\log_2 \sqrt{2x^2+1} - \log_2 x)$$

$$= \lim_{x \rightarrow \infty} \log_2 \frac{\sqrt{2x^2+1}}{x}$$

$$= \lim_{x \rightarrow \infty} \log_2 \sqrt{2 + \frac{1}{x^2}}$$

$$= \log_2 \sqrt{2}$$

$$= \frac{1}{2}$$

$$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M$$

$$\lim_{x \rightarrow 2} (\log_3 |x^2+5x-14| - \log_3 |x-2|)$$

$$= \lim_{x \rightarrow 2} \log_3 \left| \frac{x^2+5x-14}{x-2} \right|$$

$$= \lim_{x \rightarrow 2} \log_3 \left| \frac{(x+7)(x-2)}{x-2} \right|$$

$$= \lim_{x \rightarrow 2} \log_3 |x+7|$$

$$= \log_3 9$$

$$= 2$$

$$\lim_{x \rightarrow \infty} \log_2 (x+2 - \sqrt{x^2+3x+4})$$

$$= \lim_{x \rightarrow \infty} \log_2 \frac{(x+2 - \sqrt{x^2+3x+4})(x+2 + \sqrt{x^2+3x+4})}{x+2 + \sqrt{x^2+3x+4}}$$

$$= \lim_{x \rightarrow \infty} \log_2 \frac{x}{x+2 + \sqrt{x^2+3x+4}}$$

$$= \lim_{x \rightarrow \infty} \log_2 \frac{1}{1 + \frac{2}{x} + \sqrt{1 + \frac{3}{x} + \frac{4}{x^2}}}$$

$$\log_2 \frac{1}{2}$$

$$= -1$$

$$\log_a \frac{1}{N} = -\log_a N$$

Considering

$$x+2 - \sqrt{x^2+3x+4} = \frac{x+2 - \sqrt{x^2+3x+4}}{1}$$

then multiplying the numerator and the denominator of the antilogarithm

by $x+2 + \sqrt{x^2+3x+4}$

$$\sqrt[n]{M} = \frac{1}{n} \log_a M, \quad \log_a \frac{1}{N} = -\log_a N$$

Limits of Functions 2

Name _____

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Ex.

Find the cubic function $f(x)$ which satisfies $\lim_{x \rightarrow \infty} \frac{f(x) - 2x^3 + 3}{x^2} = 4$
 and $\lim_{x \rightarrow 0} \frac{f(x) - 5}{x} = 3 \dots \textcircled{2}$.

[Sol] Let $f(x) = ax^3 + bx^2 + cx + d$ ($a \neq 0$).

From $\textcircled{1}$,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x) - 2x^3 + 3}{x^2} &= \lim_{x \rightarrow \infty} \frac{(a-2)x^3 + bx^2 + cx + d + 3}{x^2} \\ &= \lim_{x \rightarrow \infty} \left[(a-2)x + b + \frac{c}{x} + \frac{d+3}{x^2} \right] \\ &= 4 \end{aligned}$$

Therefore, the values must be $a=2$ and $b=4$.

$$\therefore f(x) = 2x^3 + 4x^2 + cx + d$$

$$\text{From } \textcircled{2}, \lim_{x \rightarrow 0} \frac{2x^3 + 4x^2 + cx + d - 5}{x} = 3 \dots \textcircled{3}$$

Since $\lim_{x \rightarrow 0} x = 0$,

$$\lim_{x \rightarrow 0} (2x^3 + 4x^2 + cx + d - 5) = 0, \text{ i.e. } d - 5 = 0 \quad \leftarrow \text{N108}$$

$$\therefore d = \boxed{5}$$

Then,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2x^3 + 4x^2 + cx + d - 5}{x} &= \lim_{x \rightarrow 0} \frac{2x^3 + 4x^2 + cx}{x} \\ &= \lim_{x \rightarrow 0} (2x^2 + 4x + c) \\ &= c \dots \textcircled{4} \end{aligned}$$

$$\text{From } \textcircled{3} \text{ and } \textcircled{4}, c = \boxed{3}$$

$$\therefore f(x) = \boxed{2x^3 + 4x^2 + 3x + 5}$$

If $a \neq 2$,

$\lim_{x \rightarrow \infty} \frac{f(x) - 2x^3}{x^2}$
 does not converge

1. Find the cubic function $f(x)$ which satisfies $\lim_{x \rightarrow \infty} \frac{f(x) - x^3}{x^2} = 1$...①
and $\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = -3$...②.

[Sol] Let $f(x) = ax^3 + bx^2 + cx + d$ ($a \neq 0$).

From ①,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x) - x^3}{x^2} &= \lim_{x \rightarrow \infty} \frac{(a-1)x^3 + bx^2 + cx + d}{x^2} \\ &= \lim_{x \rightarrow \infty} \left[(a-1)x + b + \frac{c}{x} + \frac{d}{x^2} \right] \\ &= 1\end{aligned}$$

Therefore, the values must be $a=1$ and $b=1$.

$$\therefore f(x) = x^3 + x^2 + cx + d$$

$$\text{From ②, } \lim_{x \rightarrow 1} \frac{x^3 + x^2 + cx + d}{x-1} = -3 \quad \dots \text{③}$$

$$\text{Since } \lim_{x \rightarrow 1} (x-1) = 0,$$

$$\lim_{x \rightarrow 1} (x^3 + x^2 + cx + d) = 0, \text{ i.e. } 2 + c + d = 0$$

$$\therefore d = -c - 2 \quad \dots \text{④}$$

Then,

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^3 + x^2 + cx + d}{x-1} &= \lim_{x \rightarrow 1} \frac{x^3 + x^2 + cx - c - 2}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(x^2 + 2x + c + 2)}{x-1} \\ &= \lim_{x \rightarrow 1} (x^2 + 2x + c + 2) \\ &= c + 5 \quad \dots \text{⑤}\end{aligned}$$

The Factor
Theorem
(J172)

$$\text{From ① and ⑤, } c + 5 = -3; \text{ therefore, } c = -8 \quad \dots \text{⑥}$$

$$\text{From ④ and ⑥, } d = 6$$

$$\therefore f(x) = x^3 + x^2 - 8x + 6$$

Limits of Functions 2

Name _____

Date / /

Time : to :

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1. Find the constants a and b which satisfy the following equation.

$$\lim_{x \rightarrow \infty} [\sqrt{4x^2 - 12x + 1} - (ax + b)] = 0 \quad \cdots \textcircled{1}$$

[Sol] When $a \leq 0$, $\lim_{x \rightarrow \infty} [\sqrt{4x^2 - 12x + 1} - (ax + b)] = \infty$; therefore,

it does not satisfy $\textcircled{1}$.

$$\therefore a > 0 \quad \cdots \textcircled{2}$$

Then,

$$\begin{aligned} & \lim_{x \rightarrow \infty} [\sqrt{4x^2 - 12x + 1} - (ax + b)] \\ &= \lim_{x \rightarrow \infty} \frac{[\sqrt{4x^2 - 12x + 1} - (ax + b)][\sqrt{4x^2 - 12x + 1} + (ax + b)]}{\sqrt{4x^2 - 12x + 1} + (ax + b)} \\ &= \lim_{x \rightarrow \infty} \frac{(4 - a^2)x^2 - (12 + 2ab)x + 1 - b^2}{\sqrt{4x^2 - 12x + 1} + (ax + b)} \\ &= \lim_{x \rightarrow \infty} \frac{(4 - a^2)x - (12 + 2ab) + \frac{1 - b^2}{x}}{\sqrt{4 - \frac{12}{x} + \frac{1}{x^2}} + a + \frac{b}{x}} \end{aligned}$$

Considering

$$\frac{\sqrt{4x^2 - 12x + 1} - (ax + b)}{\sqrt{4x^2 - 12x + 1} + (ax + b)} = \frac{\sqrt{4x^2 - 12x + 1} - (ax + b)}{1}$$

then multiplying the numerator and the denominator by $\sqrt{4x^2 - 12x + 1} + (ax + b)$

From $\textcircled{1}$,

$$\begin{cases} 4 - a^2 = 0 & \cdots \textcircled{3} \\ 12 + 2ab = 0 & \cdots \textcircled{4} \end{cases}$$

If $4 - a^2 \neq 0$,

$\lim_{x \rightarrow \infty} [\sqrt{4x^2 - 12x + 1} - (ax + b)]$ does not converge.

From $\textcircled{2}$ and $\textcircled{3}$, $a = 2 \quad \cdots \textcircled{5}$

From $\textcircled{4}$ and $\textcircled{5}$, $b = -3$

$$\therefore a = 2, b = -3$$

2. Let l be line $y = ax$ ($a > 0$) and P be the point on curve $y = x^2$ ($0 \leq x \leq a$) such that the distance to l is maximized.

(1) Find the coordinates of point $P(p, p^2)$.

[Sol] Let d be the distance from point (t, t^2) ($0 \leq t \leq a$) on curve $y = x^2$ to line $y = ax$.

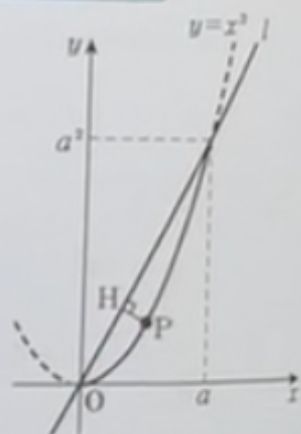
$$d = \frac{|at - t^2|}{\sqrt{a^2 + (-1)^2}}$$

$$= \frac{at - t^2}{\sqrt{a^2 + 1}}$$

$$= -\frac{1}{\sqrt{a^2 + 1}} \left[\left(t - \frac{a}{2} \right)^2 - \frac{a^2}{4} \right]$$

Distance from a Point to a Line (M25)

Since $0 \leq t \leq a$,
 $|at - t^2| = at - t^2$



Therefore, when $t = \frac{a}{2}$, d is maximized.

$$\therefore p = \frac{a}{2}$$

$$\therefore P\left(\frac{a}{2}, \frac{a^2}{4}\right)$$

- (2) Let $H(h, ah)$ be the point of intersection of the perpendicular from P to l and l . Find $\lim_{a \rightarrow \infty} \frac{h}{p}$.

[Sol] From (1), the equation of line PH is

$$y - \frac{a^2}{4} = -\frac{1}{a} \left(x - \frac{a}{2} \right)$$

Since lines PH and l are perpendicular,
the slope of line PH is $-\frac{1}{a}$.

Since this line passes through point $H(h, ah)$,

$$ah - \frac{a^2}{4} = -\frac{1}{a} \left(h - \frac{a}{2} \right)$$

$$\therefore h = \frac{a(a^2 + 2)}{4(a^2 + 1)}$$

Therefore,

$$\lim_{a \rightarrow \infty} \frac{h}{p} = \lim_{a \rightarrow \infty} \frac{\frac{a(a^2 + 2)}{4(a^2 + 1)}}{\frac{a}{2}} = \lim_{a \rightarrow \infty} \frac{a^2 + 2}{2(a^2 + 1)} = \lim_{a \rightarrow \infty} \frac{1 + \frac{2}{a^2}}{2\left(1 + \frac{1}{a^2}\right)} = \frac{1}{2}$$

Limits of Functions 2

Name _____

Date / /

Time : to :

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Find the following limits.

$$(1) \lim_{x \rightarrow \infty} \frac{2x+3}{\sqrt{x^2-1}+3} = \lim_{x \rightarrow \infty} \frac{2+\frac{3}{x}}{\sqrt{1-\frac{1}{x^2}+\frac{3}{x}}} = 2$$

$$\begin{aligned}
 (2) \lim_{x \rightarrow \infty} (\sqrt{4x^2+x}-2x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{4x^2+x}-2x)(\sqrt{4x^2+x}+2x)}{\sqrt{4x^2+x}+2x} \\
 &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2+x}+2x} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{4+\frac{1}{x}}+2} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$(3) \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+x}+2x}$$

[Sol] Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+x}+2x} &= \lim_{t \rightarrow \infty} \frac{-t}{\sqrt{t^2-t}-2t} \\
 &= \lim_{t \rightarrow \infty} \frac{-1}{\sqrt{1-\frac{1}{t}}-2} \\
 &= 1
 \end{aligned}$$

N120b

$$(4) \lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} \quad (a > 0)$$

[Sol] (i) When $a > 1$,

$$\lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{a}}{\left(\frac{1}{a}\right)^x + 1} = \frac{1}{a}$$

(ii) When $a = 1$,

$$\lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} = \frac{1}{2}$$

(iii) When $0 < a < 1$,

$$\lim_{x \rightarrow \infty} \frac{a^{x-1}}{1+a^x} = 0$$

$$(5) \lim_{x \rightarrow \infty} [\log_3(9x-1) - \log_3(x+1)]$$

➡ N117

$$= \lim_{x \rightarrow \infty} \log_3 \frac{9x-1}{x+1}$$

$$= \lim_{x \rightarrow \infty} \log_3 \frac{9 - \frac{1}{x}}{1 + \frac{1}{x}}$$

$$= \log_3 9$$

$$= 2$$

Limits of Trigonometric Functions

Name _____

Date / /

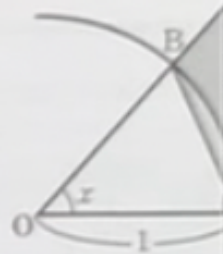
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Prove the equation $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

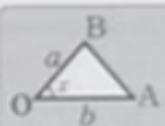
[Sol] (i) When $0 < x < \frac{\pi}{2}$, ← Considering $\lim_{x \rightarrow 0^+} \frac{\sin x}{x}$

consider sector OAB with radius 1 and central angle x as shown in the diagram. AT is a tangent to the circle at point A.



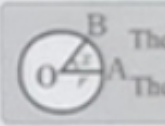
As for the areas,

$$\triangle OAB = \frac{1}{2} \sin x \quad \leftarrow$$



The area of $\triangle OAB$ is $\frac{1}{2}ab \sin x$

$$\text{Sector OAB} = \frac{1}{2}x \quad \leftarrow$$



The area of a circle with radius r is πr^2 .
The area of sector OAB is $\pi r^2 \times \frac{x}{2\pi} = \frac{1}{2}r^2 x$

$$\triangle OAT = \frac{1}{2} \tan x \quad \leftarrow$$

$$AT = OA \tan x = \tan x$$

Also, since $\triangle OAB < \text{sector OAB} < \triangle OAT$,

$$\frac{1}{2} \sin x < \frac{1}{2}x < \frac{1}{2} \tan x, \text{ i.e. } \sin x < x < \tan x$$

Since $\sin x > 0$, dividing all terms by $\sin x$,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \leftarrow$$

$$\frac{\tan x}{\sin x} = \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{1}{\cos x}$$

Since all the terms are positive,

$$1 > \frac{\sin x}{x} > \cos x \quad \leftarrow$$

When $0 < a < b < c$, $\frac{1}{a} > \frac{1}{b}$

$$\text{Since } \lim_{x \rightarrow 0^+} \cos x = 1, \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \quad \leftarrow$$

Limits of Functions
Their Relationships
(Refer to side)

(ii) When $-\frac{\pi}{2} < x < 0$, ← Considering $\lim_{x \rightarrow 0^-} \frac{\sin x}{x}$

let $x = -\theta$. As $x \rightarrow 0^-$, $\theta \rightarrow 0^+$; therefore, from (i),

$$\lim_{x \rightarrow 0^-} \frac{\sin x}{x} = \lim_{\theta \rightarrow 0^+} \frac{\sin(-\theta)}{-\theta} = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1 \quad \leftarrow \sin(-\theta) = -\sin \theta$$

From (i) and (ii), $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\theta \rightarrow 0^+ \quad \frac{\sin \theta}{\theta} \rightarrow 1 \quad \frac{\sin(-\theta)}{-\theta} \rightarrow 1 \quad \frac{\sin x}{x} \rightarrow 1 \quad \frac{\sin(-x)}{-x} \rightarrow 1$$

N121b

From the result on side a, the following equation is true.

Limit of $\frac{\sin x}{x}$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Find the following limits.

Ex.

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot 5 = 1 \cdot 5 = 5$$

Rearranging the denominator into $5x$, then rewriting as $\frac{\sin \bullet}{\bullet}$

$$(1) \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot 2 = 1 \cdot 2 = 2$$

$$2) \lim_{x \rightarrow 0} \frac{\sin(-3x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(-3x)}{-3x} \cdot (-3) = 1 \cdot (-3) = -3$$

$$3) \lim_{x \rightarrow 0} \frac{\sin 2x}{3x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}$$

s for $\ast$ on side a, the following statements are true.

Limits of Functions and Their Relationships

For all x close to a , when $f(x) \leq g(x)$,

if $\lim_{x \rightarrow a} f(x) = \alpha$ and $\lim_{x \rightarrow a} g(x) = \beta$, then $\alpha \leq \beta$

if $\lim_{x \rightarrow a} f(x) = \infty$, then $\lim_{x \rightarrow a} g(x) = \infty$

For all x close to a , when $f(x) \leq h(x) \leq g(x)$,

if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \alpha$, then $\lim_{x \rightarrow a} h(x) = \alpha$

ie when $f(x) < g(x)$, and statement 2 is also true

$f(x) < h(x) \leq g(x)$, $f(x) < h(x) < g(x)$.

Limits of Trigonometric Functions

Name _____

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Find the following limits.

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 0} \frac{x}{\sin 4x} &= \lim_{x \rightarrow 0} \frac{4x}{\sin 4x} \cdot \frac{1}{4} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin 4x}{4x}} \cdot \frac{1}{4} \\
 &= \frac{1}{1} \cdot \frac{1}{4} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow 0} \frac{2x}{\sin 3x} &= \lim_{x \rightarrow 0} \frac{3x}{\sin 3x} \cdot \frac{2}{3} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin 3x}{3x}} \cdot \frac{2}{3} \\
 &= \frac{1}{1} \cdot \frac{2}{3} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \frac{2x}{\sin 2x} \cdot \frac{3}{2} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \frac{1}{\frac{\sin 2x}{2x}} \cdot \frac{3}{2} \\
 &= 1 \cdot \frac{1}{1} \cdot \frac{3}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

Since $\lim_{x \rightarrow 0} \frac{x}{\sin x} = \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} = 1$, it is possible to solve by using $\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$.

Ex.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\tan x}{x} &= \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x} && \leftarrow \tan x = \frac{\sin x}{\cos x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} && \leftarrow \lim_{x \rightarrow 0} \cos x = 1 \\
 &= 1 \cdot \frac{1}{1} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \lim_{x \rightarrow 0} \frac{\tan 4x}{x} &= \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\sin 4x}{\cos 4x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{4}{\cos 4x} \\
 &= 1 \cdot \frac{4}{1} \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 2x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{\frac{\sin 2x}{\cos 2x}} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{3x} \cdot 3}{\frac{\sin 2x}{2x} \cdot \frac{2}{\cos 2x}} \\
 &= \frac{1 \cdot 3}{1 \cdot \frac{2}{1}} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \lim_{x \rightarrow 0} \frac{\tan^2 x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{1}{1 - \cos x} \cdot \frac{\sin^2 x}{\cos^2 x} && \leftarrow \sin^2 x = 1 - \cos^2 x \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{(1 - \cos x) \cos^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{(1 + \cos x)(1 - \cos x)}{(1 - \cos x) \cos^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + \cos x}{\cos^2 x} \\
 &= \frac{1 + 1}{1^2} \\
 &= 2
 \end{aligned}$$

Limits of Trigonometric Functions

Name _____

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Find the following limits.

Ex.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin 3x}{x + \sin x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{x}}{1 + \frac{\sin x}{x}} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{3x} \cdot 3}{1 + \frac{\sin x}{x}} \\
 &= \frac{1 \cdot 3}{1 + 1} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 0} \frac{\sin x}{x + \sin x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x}}{1 + \frac{\sin x}{x}} \\
 &= \frac{1}{1 + 1} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{x^2 + 2x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{x}}{x + 2} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{2x} \cdot 2}{x + 2} \\
 &= \frac{1 \cdot 2}{0 + 2} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 0} \frac{x - \sin 2x}{\sin 3x} &= \lim_{x \rightarrow 0} \frac{1 - \frac{\sin 2x}{x}}{\frac{\sin 3x}{x}} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \frac{\sin 2x}{2x} \cdot 2}{\frac{\sin 3x}{3x} \cdot 3} \\
 &= \frac{1 - 1 \cdot 2}{1 \cdot 3} \\
 &= -\frac{1}{3}
 \end{aligned}$$

Alternative Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x - \sin 2x}{\sin 3x} &= \lim_{x \rightarrow 0} \left(\frac{x}{\sin 3x} - \frac{\sin 2x}{\sin 3x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{3x}{\sin 3x} \cdot \frac{1}{3} - \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2}{3} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\frac{\sin 3x}{3x}} \cdot \frac{1}{3} - \frac{\sin 2x}{2x} \cdot \frac{1}{\frac{\sin 3x}{3x}} \cdot \frac{2}{3} \right) \\
 &= \frac{1}{1} \cdot \frac{1}{3} - 1 \cdot \frac{1}{1} \cdot \frac{2}{3} \\
 &= -\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \lim_{x \rightarrow 0} \frac{\sin x}{x + \tan x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x}}{1 + \frac{1}{x} \cdot \frac{\sin x}{\cos x}} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x}}{1 + \frac{\sin x}{x} \cdot \frac{1}{\cos x}} \\
 &= \frac{1}{1 + 1 \cdot \frac{1}{1}} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \lim_{x \rightarrow 0} \frac{\sin x - \tan 2x}{x} &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} - \frac{1}{x} \cdot \frac{\sin 2x}{\cos 2x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} - \frac{\sin 2x}{2x} \cdot \frac{2}{\cos 2x} \right) \\
 &= 1 - 1 \cdot \frac{2}{1} \\
 &= -1
 \end{aligned}$$

Limits of Trigonometric Functions

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Find the following limits.

Ex.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{1 + \cos x} \\
 &= \frac{1}{2}
 \end{aligned}$$

← Multiplying the numerator and the denominator by $1 + \cos x$

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sin x(1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{x \sin x(1 + \cos x)}{1 - \cos^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{x \sin x(1 + \cos x)}{\sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + \cos x}{\frac{\sin x}{x}} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow 0} \frac{\sin^3 x}{x(1 - \cos x)} &= \lim_{x \rightarrow 0} \frac{\sin^3 x(1 + \cos x)}{x(1 - \cos x)(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^3 x(1 + \cos x)}{x(1 - \cos^2 x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^3 x(1 + \cos x)}{x \sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot (1 + \cos x) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{3x^2} &= \lim_{x \rightarrow 0} \frac{(1 - \cos 3x)(1 + \cos 3x)}{3x^2(1 + \cos 3x)} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 3x}{3x^2(1 + \cos 3x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 3x}{3x^2(1 + \cos 3x)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x} \right)^2 \cdot \frac{3}{1 + \cos 3x} \\
 &= \frac{3}{2}
 \end{aligned}$$

Alternative Solution

$$\begin{aligned}
 &\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{3}{2}x}{3x^2} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin \frac{3}{2}x}{\frac{3}{2}x} \right)^2 \cdot \frac{3}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{x^3 \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)(1 + \cos x)}{x^3 \cos x(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos^2 x)}{x^3 \cos x(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^3 x}{x^3 \cos x(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^3 \cdot \frac{1}{\cos x(1 + \cos x)} \\
 &= \frac{1}{2}
 \end{aligned}$$

Alternative Solution

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{x^3 \cos x} = \lim_{x \rightarrow 0} \frac{\sin x \cdot 2 \sin^2 \frac{x}{2}}{x^3 \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \cdot \frac{1}{2 \cos x} = \frac{1}{2}$$

These can also be solved by using the Half-Angle Formula,

$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$ (M164). In the case of Ex.

$$\lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \cdot \frac{1}{2} = \frac{1}{2}$$

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Find the following limits.

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 0} \frac{2 \tan x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \frac{2 \sin x (\sqrt{1+x}+1)}{\cos x (\sqrt{1+x}-1)(\sqrt{1+x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin x (\sqrt{1+x}+1)}{x \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{2(\sqrt{1+x}+1)}{\cos x} \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1+x \sin x}-1}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x \sin x}-1)(\sqrt{1+x \sin x}+1)}{x^2(\sqrt{1+x \sin x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{x \sin x}{x^2(\sqrt{1+x \sin x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\sqrt{1+x \sin x}+1} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 0} \frac{\sqrt{2-\cos x}-1}{3x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{2-\cos x}-1)(\sqrt{2-\cos x}+1)}{3x^2(\sqrt{2-\cos x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{1-\cos x}{3x^2(\sqrt{2-\cos x}+1)} \\
 &= \lim_{x \rightarrow 0} \frac{(1-\cos x)(1+\cos x)}{3x^2(\sqrt{2-\cos x}+1)(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{3x^2(\sqrt{2-\cos x}+1)(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{3x^2(\sqrt{2-\cos x}+1)(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{3(\sqrt{2-\cos x}+1)(1+\cos x)} \\
 &= \frac{1}{12}
 \end{aligned}$$

Alternative Solution

$$\lim_{x \rightarrow 0} \frac{\sqrt{2-\cos x}-1}{3x^2} = \lim_{x \rightarrow 0} \frac{1-\cos x}{3x^2(\sqrt{2-\cos x}+1)} = \lim_{x \rightarrow 0} \frac{2\sin^2 \frac{x}{2}}{3x^2(\sqrt{2-\cos x}+1)} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \cdot \frac{1}{6(\sqrt{2-\cos x}+1)} = \frac{1}{12}$$

$$\begin{aligned}
 (4) \quad \lim_{x \rightarrow 0} \frac{1-\cos x}{\sqrt{1+x^2}-\sqrt{1-x^2}} &= \lim_{x \rightarrow 0} \frac{(1-\cos x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})(\sqrt{1+x^2}+\sqrt{1-x^2})} \\
 &= \lim_{x \rightarrow 0} \frac{(1-\cos x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2} \\
 &= \lim_{x \rightarrow 0} \frac{(1-\cos x)(1+\cos x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{(1-\cos^2 x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2(1+\cos x)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{2(1+\cos x)} = \frac{1}{2}
 \end{aligned}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin^2 \frac{x}{2}(\sqrt{1+x^2}+\sqrt{1-x^2})}{(1+x^2-\sqrt{1-x^2})(\sqrt{1+x^2}+\sqrt{1-x^2})} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \cdot \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{4} = \frac{1}{2}$$

Limits of Trigonometric Functions

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Find the following limits.

Ex.

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$$

[Sol] Let $\frac{1}{x} = \theta$. As $x \rightarrow \infty$, $\theta \rightarrow 0^+$

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} x \sin \frac{1}{x} &= \lim_{\theta \rightarrow 0^+} \frac{1}{\theta} \sin \theta \\ &= \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} \\ &= 1 \end{aligned}$$

$$(1) \lim_{x \rightarrow \infty} x \tan \frac{1}{x}$$

[Sol] Let $\frac{1}{x} = \theta$. As $x \rightarrow \infty$, $\theta \rightarrow 0^+$

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} x \tan \frac{1}{x} &= \lim_{\theta \rightarrow 0^+} \frac{1}{\theta} \tan \theta \\ &= \lim_{\theta \rightarrow 0^+} \frac{1}{\theta} \cdot \frac{\sin \theta}{\cos \theta} \\ &= \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta} \\ &= 1 \end{aligned}$$

$$(2) \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{\sin x}$$

[Sol] Let $\sin x = \theta$. As $x \rightarrow 0$, $\theta \rightarrow 0$ ←

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{\sin x} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \\ &= 1 \end{aligned}$$

As $x \rightarrow 0^+$, $\theta \rightarrow 0^+$ As $x \rightarrow 0^-$, $\theta \rightarrow 0^-$ Therefore, use $\theta \rightarrow 0$ instead of $\theta \rightarrow 0^+$ or $\theta \rightarrow 0^-$.

Ex.

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{2\pi - 8x}{\sin(4x - \pi)}$$

[Sol] Let $x - \frac{\pi}{4} = \theta$. $x = \theta + \frac{\pi}{4}$

As $x \rightarrow \frac{\pi}{4}$, $\theta \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{4}} \frac{2\pi - 8x}{\sin(4x - \pi)} &= \lim_{\theta \rightarrow 0} \frac{-8\theta}{\sin 4\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{-2}{\frac{\sin 4\theta}{4\theta}} \\ &= -2 \end{aligned}$$



$$\begin{aligned} 2\pi - 8x &= 2\pi - 8\left(\theta + \frac{\pi}{4}\right) = -8\theta \\ 4x - \pi &= 4\left(\theta + \frac{\pi}{4}\right) - \pi = 4\theta \end{aligned}$$

(3) $\lim_{x \rightarrow \frac{\pi}{6}} \frac{6x - \pi}{\sin\left(x - \frac{\pi}{6}\right)}$

[Sol] Let $x - \frac{\pi}{6} = \theta$. $x = \theta + \frac{\pi}{6}$

As $x \rightarrow \frac{\pi}{6}$, $\theta \rightarrow 0$

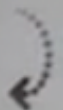
$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{6}} \frac{6x - \pi}{\sin\left(x - \frac{\pi}{6}\right)} &= \lim_{\theta \rightarrow 0} \frac{6\theta}{\sin \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{6}{\frac{\sin \theta}{\theta}} \\ &= 6 \end{aligned}$$

(4) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{2x - \pi}$

[Sol] Let $x - \frac{\pi}{2} = \theta$. $x = \theta + \frac{\pi}{2}$

As $x \rightarrow \frac{\pi}{2}$, $\theta \rightarrow 0$

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{2x - \pi} &= \lim_{\theta \rightarrow 0} \frac{\cos\left(\theta + \frac{\pi}{2}\right)}{2\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{-\sin \theta}{2\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \left(-\frac{1}{2}\right) \\ &= -\frac{1}{2} \end{aligned}$$



$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta$$

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Find the following limits.

Ex.

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x}$$

$$|ab| = |a||b|$$

[Sol] Since $0 \leq \left| \sin \frac{1}{x} \right| \leq 1$, $0 \leq \left| x \sin \frac{1}{x} \right| = |x| \left| \sin \frac{1}{x} \right| \leq |x|$ ←

Then, since $\lim_{x \rightarrow 0} |x| = 0$,

$$\lim_{x \rightarrow 0} \left| x \sin \frac{1}{x} \right| = 0$$

Limits of Functions and
Their Relationships (N121)

$$\therefore \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

If $\lim_{x \rightarrow a} |f(x)| = 0$, then $\lim_{x \rightarrow a} f(x) = 0$

(1) $\lim_{x \rightarrow 0} x \cos \frac{1}{x}$

[Sol] Since $0 \leq \left| \cos \frac{1}{x} \right| \leq 1$, $0 \leq \left| x \cos \frac{1}{x} \right| = |x| \left| \cos \frac{1}{x} \right| \leq |x|$

Then, since $\lim_{x \rightarrow 0} |x| = 0$,

$$\lim_{x \rightarrow 0} \left| x \cos \frac{1}{x} \right| = 0$$

$$\therefore \lim_{x \rightarrow 0} x \cos \frac{1}{x} = 0$$

(2) $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$

[Sol] Since $0 \leq \left| \sin \frac{1}{x} \right| \leq 1$, $0 \leq \left| x^2 \sin \frac{1}{x} \right| = x^2 \left| \sin \frac{1}{x} \right| \leq x^2$

Then, since $\lim_{x \rightarrow 0} x^2 = 0$,

$$\lim_{x \rightarrow 0} \left| x^2 \sin \frac{1}{x} \right| = 0$$

$$\therefore \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

N127b

$$(3) \lim_{x \rightarrow \infty} \frac{\cos x}{x}$$

$$[\text{Sol}] \text{ Since } 0 \leq |\cos x| \leq 1, \quad 0 \leq \left| \frac{\cos x}{x} \right| = \left| \frac{1}{x} \right| |\cos x| \leq \left| \frac{1}{x} \right|$$

$$\text{Then, since } \lim_{x \rightarrow \infty} \left| \frac{1}{x} \right| = 0,$$

$$\lim_{x \rightarrow \infty} \left| \frac{\cos x}{x} \right| = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$$

$$(4) \lim_{x \rightarrow \infty} \frac{x}{x + \sin x}$$

$$[\text{Sol}] \lim_{x \rightarrow \infty} \frac{x}{x + \sin x} = \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{\sin x}{x}}$$

$$\text{Since } 0 \leq |\sin x| \leq 1, \quad 0 \leq \left| \frac{\sin x}{x} \right| = \left| \frac{1}{x} \right| |\sin x| \leq \left| \frac{1}{x} \right|$$

$$\text{Then, since } \lim_{x \rightarrow \infty} \left| \frac{1}{x} \right| = 0,$$

$$\lim_{x \rightarrow \infty} \left| \frac{\sin x}{x} \right| = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{x + \sin x} = 1$$

If the absolute values are not used as in N67, the different conditions need to be considered.
In the case of **Ex.**, since $-1 \leq \sin \frac{1}{x} \leq 1$, the conditions are:

$$\text{When } x > 0, -x \leq x \sin \frac{1}{x} \leq x$$

$$\text{When } x < 0, x \leq x \sin \frac{1}{x} \leq -x$$

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1. Given sector OAB below, drop a perpendicular BH from point B to OA.

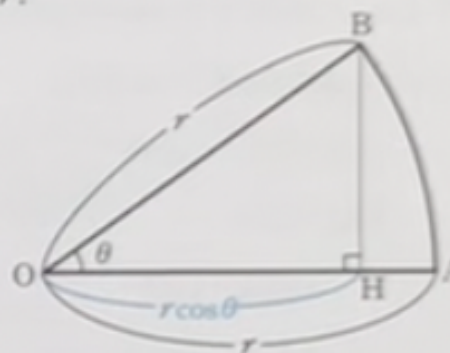
Let $\angle AOB = \theta$ ($0 < \theta < \frac{\pi}{2}$) and $OA = OB = r$.

- (1) Express AH in terms of θ .

[Sol] $AH = OA - OH$

$$= r - \boxed{r \cos \theta}$$

$$= r(\boxed{1 - \cos \theta})$$



- (2) Find $\lim_{\theta \rightarrow 0} \frac{AH}{\theta^2}$.

[Sol] $\lim_{\theta \rightarrow 0} \frac{AH}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{r(1 - \cos \theta)}{\theta^2}$

$$= \lim_{\theta \rightarrow 0} \frac{r(1 - \cos \theta)(1 + \cos \theta)}{\theta^2(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{r(1 - \cos^2 \theta)}{\theta^2(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{r \sin^2 \theta}{\theta^2(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right)^2 \cdot \frac{r}{1 + \cos \theta}$$

$$= \frac{r}{2}$$

Alternative Solution

$$\left[\lim_{\theta \rightarrow 0} \frac{AH}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{r(1 - \cos \theta)}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{2r \sin^2 \frac{\theta}{2}}{\theta^2} = \lim_{\theta \rightarrow 0} \left(\frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} \right)^2 \cdot \frac{r}{2} = \frac{r}{2} \right]$$

2. Given fixed point A on the circumference of a circle with center at point O and radius 1, drop a perpendicular PQ from another distinct point P on the circumference of the circle to the tangent at point A ($0 < \angle AOP < \frac{\pi}{2}$). Find the limit of $\frac{AQ^2}{PQ}$ when P approaches A.

[Sol] Let $\angle AOP = \theta$, and PH be a perpendicular dropped from P to OA.

$$AQ = HP = \sin \theta$$

$$PQ = HA$$

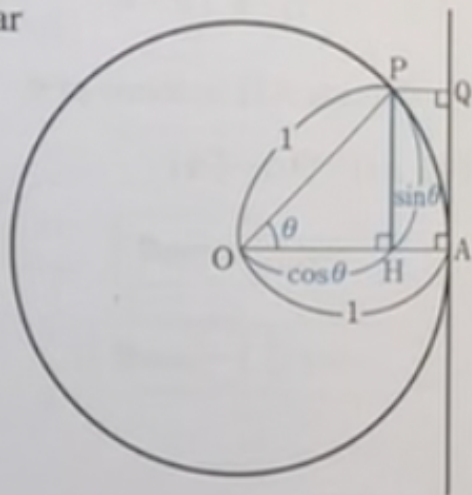
$$= OA - OH$$

$$= 1 - \cos \theta$$

When P approaches A, $\theta \rightarrow 0^+$

Therefore,

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{AQ^2}{PQ} &= \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta}{1 - \cos \theta} \\ &= \lim_{\theta \rightarrow 0^+} \frac{1 - \cos^2 \theta}{1 - \cos \theta} \\ &= \lim_{\theta \rightarrow 0^+} \frac{(1 + \cos \theta)(1 - \cos \theta)}{1 - \cos \theta} \\ &= \lim_{\theta \rightarrow 0^+} (1 + \cos \theta) \\ &= 2 \end{aligned}$$



Alternative Solution 1

$$\lim_{\theta \rightarrow 0^+} \frac{AQ^2}{PQ} = \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta}{1 - \cos \theta} = \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta}{2 \sin^2 \frac{\theta}{2}} = \lim_{\theta \rightarrow 0^+} \frac{\left(\frac{\sin \theta}{\theta}\right)^2 \cdot 2}{\left(\frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}}\right)^2} = 2$$

Alternative Solution 2

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{AQ^2}{PQ} &= \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta}{1 - \cos \theta} = \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta (1 + \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta (1 + \cos \theta)}{1 - \cos^2 \theta} = \lim_{\theta \rightarrow 0^+} \frac{\sin^2 \theta (1 + \cos \theta)}{\sin^2 \theta} \\ &= \lim_{\theta \rightarrow 0^+} (1 + \cos \theta) = 2 \end{aligned}$$

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1. Find the constants a and b which satisfy the following equation.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{ax+b}{\cos x} = \frac{1}{2} \quad \dots \textcircled{1}$$

[Sol] Since $\lim_{x \rightarrow \frac{\pi}{2}} \cos x = 0$,

$$\lim_{x \rightarrow \frac{\pi}{2}} (ax+b) = 0, \text{ i.e. } \frac{1}{2}\pi a + b = 0 \quad \leftarrow \text{N108}$$

$$\therefore b = -\frac{1}{2}\pi a \quad \dots \textcircled{2}$$

Then,

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{ax+b}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{a\left(x - \frac{\pi}{2}\right)}{\cos x}$$

$$\text{Let } x - \frac{\pi}{2} = \theta. \quad x = \theta + \frac{\pi}{2}$$

$$\text{As } x \rightarrow \frac{\pi}{2}, \theta \rightarrow 0$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}} \frac{a\left(x - \frac{\pi}{2}\right)}{\cos x} &= \lim_{\theta \rightarrow 0} \frac{a\theta}{\cos\left(\theta + \frac{\pi}{2}\right)} \\ &= \lim_{\theta \rightarrow 0} \frac{a\theta}{-\sin \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{-a}{\frac{\sin \theta}{\theta}} \\ &= -a \quad \dots \textcircled{3} \end{aligned}$$

$$\cos\left(\theta + \frac{\pi}{2}\right) = -$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{3}, -a = \frac{1}{2}; \text{ therefore, } a = -\frac{1}{2} \quad \dots \textcircled{4}$$

$$\text{From } \textcircled{2} \text{ and } \textcircled{4}, b = \frac{\pi}{4}$$

$$\therefore a = -\frac{1}{2}, \quad b = \frac{\pi}{4}$$

2. Given two points $A(2, 0)$ and $B(0, 1)$ on the coordinate plane with origin O , let $\angle AOP_n = \theta_n$ and P_n be the point internally dividing line segment AB in the ratio $1:n$ (n is a natural number and $0 < \theta_n < \frac{\pi}{2}$). Let l_n be the length of line segment AP_n . Find the limit value of $\lim_{n \rightarrow \infty} \frac{l_n}{\theta_n}$.

[Sol] Let (x_n, y_n) be the coordinates of point P_n . Since point P_n internally divides line segment AB in the ratio $1:n$,

$$x_n = \frac{n \cdot 2 + 1 \cdot 0}{1+n} = \frac{2n}{1+n}, \quad y_n = \frac{n \cdot 0 + 1 \cdot 1}{1+n} = \frac{1}{1+n} \quad \leftarrow \text{Coordinates of Internal Dividing Points (M6)}$$

$$\text{Therefore, since } OP_n = \sqrt{\left(\frac{2n}{1+n}\right)^2 + \left(\frac{1}{1+n}\right)^2} = \frac{\sqrt{4n^2+1}}{1+n},$$

$$\sin \theta_n = \frac{\frac{1}{1+n}}{\frac{\sqrt{4n^2+1}}{1+n}} = \frac{1}{\sqrt{4n^2+1}}$$

$$\text{Also, since } AB = \sqrt{5}, \quad l_n = \frac{1}{n+1} AB = \frac{\sqrt{5}}{n+1}$$

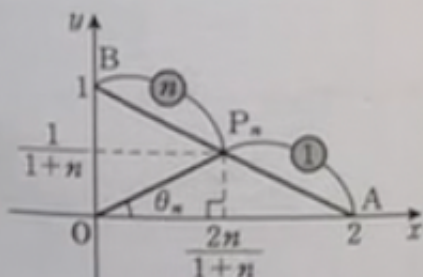
$$\therefore \lim_{n \rightarrow \infty} \frac{l_n}{\theta_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{5}}{(n+1)\theta_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{5}}{(n+1)\sin \theta_n} \cdot \frac{\sin \theta_n}{\theta_n}$$

$$\text{Then, } \lim_{n \rightarrow \infty} \frac{\sqrt{5}}{(n+1)\sin \theta_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{5}\sqrt{4n^2+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{\sqrt{5}\sqrt{4+\frac{1}{n^2}}}{1+\frac{1}{n}} = 2\sqrt{5}$$

Also, as $n \rightarrow \infty, \theta_n \rightarrow 0^+$; therefore,

$$\lim_{n \rightarrow \infty} \frac{\sin \theta_n}{\theta_n} = \lim_{\theta_n \rightarrow 0^+} \frac{\sin \theta_n}{\theta_n} = 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{l_n}{\theta_n} = 2\sqrt{5}$$



Alternative Solution

$$\text{In } \triangle OAP_n, \quad \frac{l_n}{\sin \theta_n} = \frac{2}{\sin \angle OP_n A} \quad \text{--- (1)} \quad \leftarrow \text{Sine Rule (M182)}$$

$$\text{Since } AB = \sqrt{5}, \text{ in } \triangle OBP_n, \quad \frac{\sqrt{5}-l_n}{\sin\left(\frac{\pi}{2}-\theta_n\right)} = \frac{1}{\sin(\pi-\angle OP_n A)}$$

$$\therefore \frac{\sqrt{5}-l_n}{\cos \theta_n} = \frac{1}{\sin \angle OP_n A} \quad \text{--- (2)} \quad \leftarrow \begin{matrix} \sin\left(\frac{\pi}{2}-\theta\right) = \cos \theta \\ \sin(\pi-\theta) = \sin \theta \end{matrix}$$

$$\text{From (1) and (2), } l_n = \frac{2\sqrt{5}\sin \theta_n}{2\sin \theta_n + \cos \theta_n}$$

$$\text{As } n \rightarrow \infty, \theta_n \rightarrow 0^+$$

$$\therefore \lim_{n \rightarrow \infty} \frac{l_n}{\theta_n} = \lim_{\theta_n \rightarrow 0^+} \frac{2\sqrt{5}\sin \theta_n}{\theta_n(2\sin \theta_n + \cos \theta_n)} = \lim_{\theta_n \rightarrow 0^+} \frac{\sin \theta_n}{\theta_n} \cdot \frac{2\sqrt{5}}{2\sin \theta_n + \cos \theta_n} = 2\sqrt{5}$$

Limits of Trigonometric Functions

Name _____

Date / /

Time : to :

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(mistakes) 0	—	1	—	2~

Find the following limits.

$$\begin{aligned}
 (1) \quad \lim_{x \rightarrow 0} \frac{\tan 2x}{x} &= \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\sin 2x}{\cos 2x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \frac{2}{\cos 2x} \\
 &= 2
 \end{aligned}$$

➡ N122

$$\begin{aligned}
 (2) \quad \lim_{x \rightarrow 0} \frac{\sin 4x}{x + \sin x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 4x}{x}}{1 + \frac{\sin x}{x}} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin 4x}{4x} \cdot 4}{1 + \frac{\sin x}{x}} \\
 &= 2
 \end{aligned}$$

➡ N123

$$\begin{aligned}
 (3) \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 \cos x} &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 \cos x (1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2 \cos x (1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2 \cos x (1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{\cos x (1 + \cos x)} \\
 &= \frac{1}{2}
 \end{aligned}$$

➡ N124

Alternative Solution

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 \cos x} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2 \cos x} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \cdot \frac{1}{2 \cos x} = \frac{1}{2}$$

$$(4) \lim_{x \rightarrow \frac{\pi}{6}} \frac{3\pi - 18x}{\sin(6x - \pi)}$$

➡ N126

[Sol] Let $x - \frac{\pi}{6} = \theta$. $x = \theta + \frac{\pi}{6}$

As $x \rightarrow \frac{\pi}{6}$, $\theta \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{6}} \frac{3\pi - 18x}{\sin(6x - \pi)} &= \lim_{\theta \rightarrow 0} \frac{-18\theta}{\sin 6\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{-3}{\frac{\sin 6\theta}{6\theta}} \\ &= -3 \end{aligned}$$

$$(5) \lim_{x \rightarrow \infty} \frac{\sin x}{x}$$

➡ N127

[Sol] Since $0 \leq |\sin x| \leq 1$, $0 \leq \left| \frac{\sin x}{x} \right| = \left| \frac{1}{x} \right| |\sin x| \leq \left| \frac{1}{x} \right|$

Then, since $\lim_{x \rightarrow \infty} \left| \frac{1}{x} \right| = 0$,

$$\lim_{x \rightarrow \infty} \left| \frac{\sin x}{x} \right| = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

Continuous and Discontinuous Functions

Name _____

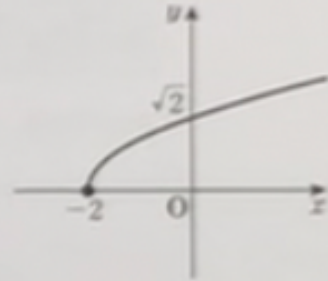
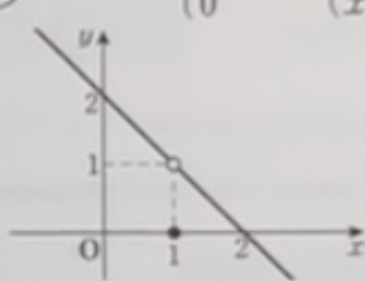
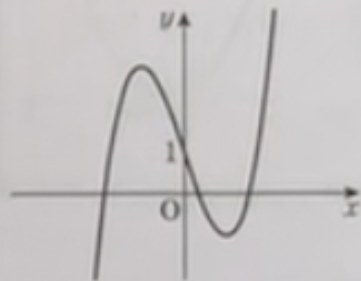
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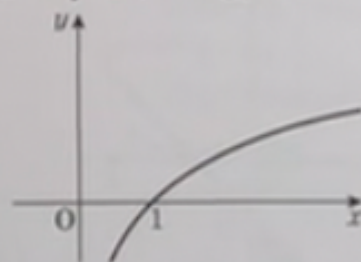
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[I] Choose two graphs that have a break(s) within the domain from (A) ~ (F) and circle the letters.

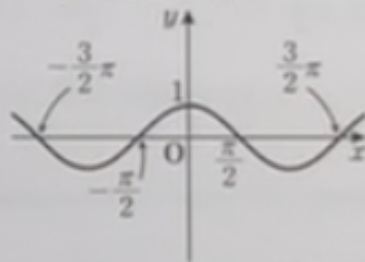
(A) $f(x) = x^3 - 3x + 1$ (B) $f(x) = \begin{cases} -x+2 & (x \neq 1) \\ 0 & (x = 1) \end{cases}$ (C) $f(x) = \sqrt{x+2}$



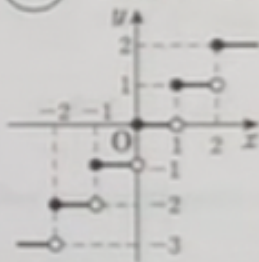
(D) $f(x) = \log_2 x$



(E) $f(x) = \cos x$



(F) $f(x) = [x]$



([x] represents the greatest integer less than or equal to x.)

Answers: (B), (F)

(The domain of (C) is $x \geq -2$, and the domain of (D) is $x > 0$. Therefore, they do not break within the domain.)

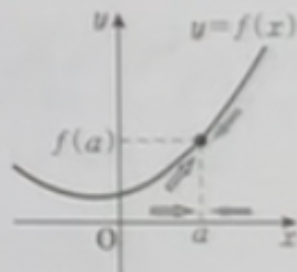
Continuous and Discontinuous Functions

Generally, the function $f(x)$ is said to be **continuous** at $x = a$ if $f(x)$ satisfies the following two conditions with respect to a , which is the value of x within the domain.

(i) $\lim_{x \rightarrow a} f(x)$ exists.

(ii) $\lim_{x \rightarrow a} f(x) = f(a)$ is true.

With these conditions, the graph of $y = f(x)$ does not break at $x = a$. If the function $f(x)$ is not continuous at $x = a$, $f(x)$ is said to be **discontinuous** at $x = a$.



As shown in [I], most of the functions introduced thus far are continuous for all values of x within the domain.

In (B), since $\lim_{x \rightarrow 1} f(x) = 1$ and $f(1) = 0$, $\lim_{x \rightarrow 1} f(x) \neq f(1)$.

In (F), since $\lim_{x \rightarrow n} f(x) = n - 1$ and $\lim_{x \rightarrow n} f(x) = n$ when n is an integer, $\lim_{x \rightarrow n} f(x)$ does not exist.

N131b

Examine if the following functions are continuous at $x=0$.

Ex.

$$f(x) = \begin{cases} x^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}$$

[Sol] $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 = 0$

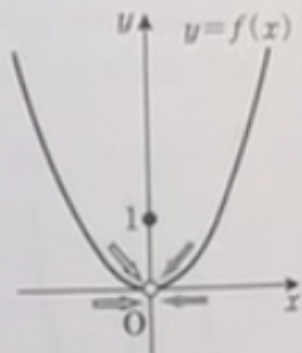
Examining if
 $\lim_{x \rightarrow a} f(x)$ exists

Also, $f(0) = 1$

Examining if
 $\lim_{x \rightarrow a} f(x) = f(a)$
is true

$\therefore \lim_{x \rightarrow 0} f(x) \neq f(0)$

Therefore, $f(x)$ is discontinuous at $x=0$.



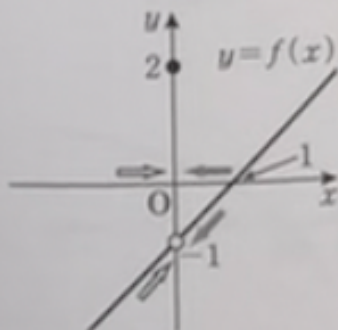
(1) $f(x) = \begin{cases} x-1 & (x \neq 0) \\ 2 & (x = 0) \end{cases}$

[Sol] $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x-1) = -1$

Also, $f(0) = 2$

$\therefore \lim_{x \rightarrow 0} f(x) \neq f(0)$

Therefore, $f(x)$ is discontinuous at $x=0$.



(2) $f(x) = \begin{cases} \frac{\sin x}{x} & (x \neq 0) \\ 1 & (x = 0) \end{cases}$

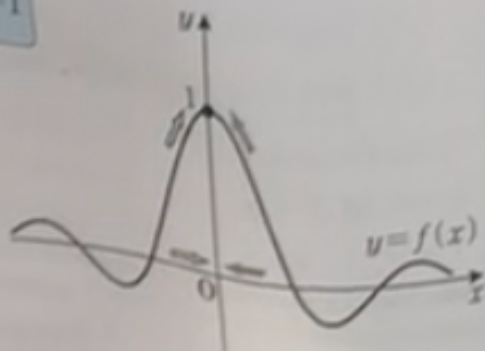
[Sol] $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Also, $f(0) = 1$

$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$

Therefore, $f(x)$ is continuous at $x=0$.



Continuous and Discontinuous Functions

Name _____

Date ____/____/____

Time ____:____ to ____:____

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Examine if the following functions are continuous at $x=0$.($[x]$ represents the greatest integer less than or equal to x .)

Ex. $f(x) = [x]$

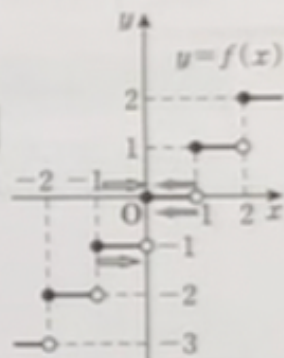
[Sol] (i) When $-1 \leq x < 0$, $[x] = -1$ Examining if
 $\lim_{x \rightarrow 0} f(x)$ exists

$$\therefore \lim_{x \rightarrow 0^-} f(x) = -1$$

(ii) When $0 \leq x < 1$, $[x] = 0$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

From (i) and (ii), $\lim_{x \rightarrow 0} f(x)$ does not exist.Therefore, $f(x)$ is discontinuous at $x=0$.

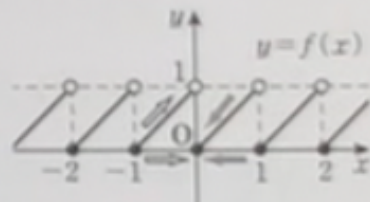
(1) $f(x) = x - [x]$

[Sol] (i) When $-1 \leq x < 0$, $[x] = -1$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} [x - (-1)] = 1$$

(ii) When $0 \leq x < 1$, $[x] = 0$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

From (i) and (ii), $\lim_{x \rightarrow 0} f(x)$ does not exist.Therefore, $f(x)$ is discontinuous at $x=0$.

N132b

(2) $f(x) = x[x]$

[Sol] (i) When $-1 \leq x < 0$, $[x] = -1$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

(ii) When $0 \leq x < 1$, $[x] = 0$

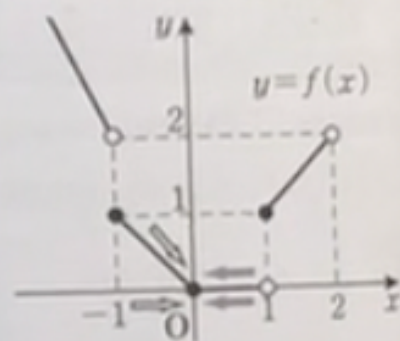
$$\therefore \lim_{x \rightarrow 0^+} f(x) = 0$$

From (i) and (ii), $\lim_{x \rightarrow 0} f(x) = 0$

Also, $f(0) = 0$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, $f(x)$ is **continuous** at $x=0$.



$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

Examining if
 $\lim_{x \rightarrow a} f(x) = f(a)$
is true

(3) $f(x) = [\cos x]$

[Sol] (i) When $-\frac{\pi}{2} \leq x < 0$, $[\cos x] = 0$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = 0$$

(ii) When $0 < x \leq \frac{\pi}{2}$, $[\cos x] = 1$

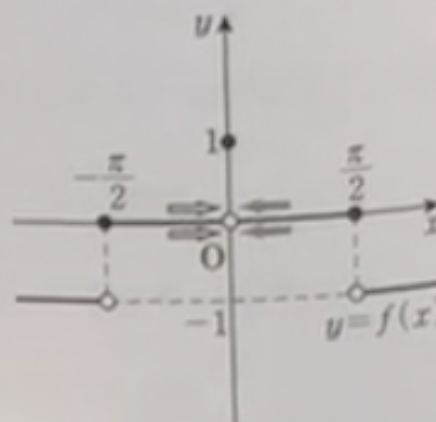
$$\therefore \lim_{x \rightarrow 0^+} f(x) = 1$$

From (i) and (ii), $\lim_{x \rightarrow 0} f(x) \neq 0$

Also, $f(0) = [\cos 0] = 1$

$$\therefore \lim_{x \rightarrow 0} f(x) \neq f(0)$$

Therefore, $f(x)$ is **discontinuous** at $x=0$.



Continuous and Discontinuous Functions

Name _____

Date / /

Time : to :

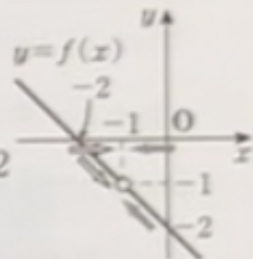
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Examine the continuity of the following functions.

Ex.

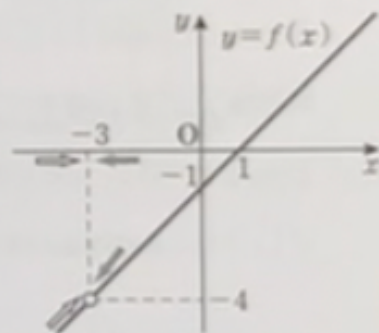
$$f(x) = -\frac{x^2 + 3x + 2}{x + 1}$$

[Sol] When $x \neq -1$, $f(x) = -\frac{(x+2)(x+1)}{x+1} = -x-2$

Therefore, $f(x)$ is continuouswhen $x < -1$ and $-1 < x$.Since $x = -1$ is not within the domain,Denominator $\neq 0$ $f(x)$ is continuous for all x , excluding the point at $x = -1$.

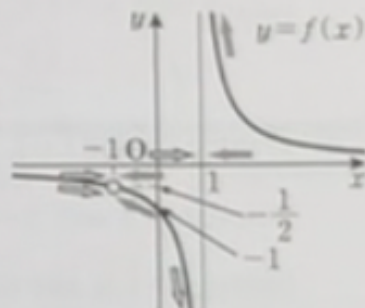
(1) $f(x) = \frac{x^2 + 2x - 3}{x + 3}$

[Sol] When $x \neq -3$, $f(x) = \frac{(x+3)(x-1)}{x+3} = x-1$

Therefore, $f(x)$ is continuouswhen $x < -3$ and $-3 < x$.Since $x = -3$ is not within the domain, $f(x)$ is continuous for all x , excluding the point at $x = -3$.

(2) $f(x) = \frac{x+1}{x^2-1}$

[Sol] When $x \neq \pm 1$, $f(x) = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}$

Therefore, $f(x)$ is continuouswhen $x < -1$, $-1 < x < 1$ and $1 < x$.Since $x = \pm 1$ is not within the domain, $f(x)$ is continuous for all x , excluding the point at $x = \pm 1$.

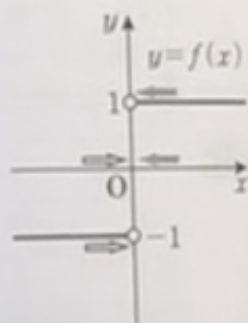
$$(3) f(x) = \frac{|x|}{x}$$

[Sol] (i) When $x > 0$, $f(x) = \frac{x}{x} = \boxed{1}$
 (ii) When $x < 0$, $f(x) = \frac{-x}{x} = \boxed{-1}$

From (i) and (ii), $f(x)$ is continuous
 when $x < 0$ and $0 < x$.

Since $x=0$ is not within the domain,

$f(x)$ is continuous for all x , excluding the point at $x=0$.



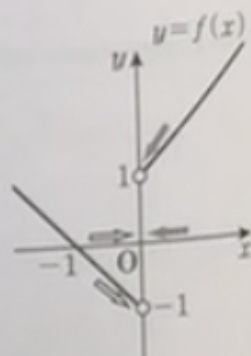
$$(4) f(x) = \frac{x^2 + x}{|x|}$$

[Sol] (i) When $x > 0$, $f(x) = \frac{x(x+1)}{x} = x+1$
 (ii) When $x < 0$, $f(x) = \frac{x(x+1)}{-x} = -x-1$

From (i) and (ii), $f(x)$ is continuous
 when $x < 0$ and $0 < x$.

Since $x=0$ is not within the domain,

$f(x)$ is continuous for all x , excluding the point at $x=0$.



$$(5) f(x) = \frac{(x-1)x^2}{|x-1|}$$

[Sol] (i) When $x-1 > 0$, i.e. $x > 1$,

$$f(x) = \frac{(x-1)x^2}{x-1} = x^2$$

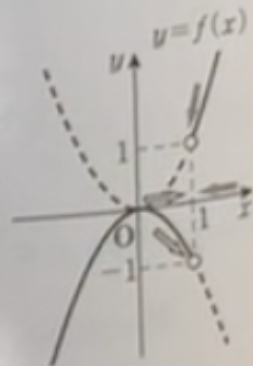
(ii) When $x-1 < 0$, i.e. $x < 1$,

$$f(x) = \frac{(x-1)x^2}{-(x-1)} = -x^2$$

From (i) and (ii), $f(x)$ is continuous
 when $x < 1$ and $1 < x$.

Since $x=1$ is not within the domain,

$f(x)$ is continuous for all x , excluding the point at $x=1$.



Continuous and Discontinuous Functions

Name _____

Date _____ / _____ / _____

Time _____ : _____ to _____ : _____

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Graph each given function and find x at which $f(x)$ is discontinuous.**Ex.**

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n}}{1 + x^{2n}}$$

[Sol] (i) When $|x| < 1$,

$$f(x) = \boxed{0}$$

$$\leftarrow \lim_{n \rightarrow \infty} x^{2n} = 0$$

(ii) When $x = \pm 1$,

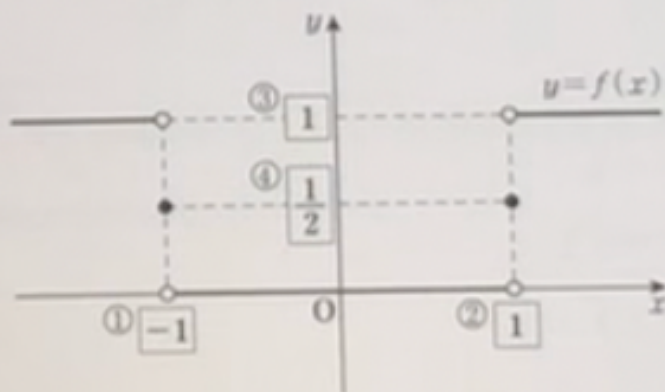
$$f(x) = \boxed{\frac{1}{2}}$$

$$\leftarrow \lim_{n \rightarrow \infty} x^{2n} = 1$$

(iii) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{x^{2n}} + 1} = \boxed{1}$$

$$\leftarrow \lim_{n \rightarrow \infty} \frac{1}{x^{2n}} = 0$$

From (i) ~ (iii), the graph of $y = f(x)$ is as shown in the diagram.Also, $f(x)$ is discontinuous at $x = \boxed{\pm 1}$.Answers: 0, $\frac{1}{2}$, 1, ± 1 , (on the graph) ① -1, ② 1, ③ 1, ④ $\frac{1}{2}$

N134b

$$(1) f(x) = \lim_{n \rightarrow \infty} \frac{|x|^n - 1}{|x|^n + 1}$$

[Sol] (i) When $|x| < 1$,

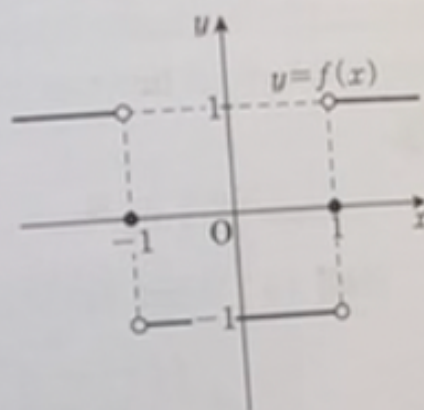
$$f(x) = -1$$

(ii) When $x = \pm 1$,

$$f(x) = 0$$

(iii) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{|x|^n}}{1 + \frac{1}{|x|^n}} = 1$$



From (i) ~ (iii), the graph of $y = f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x = \pm 1$.

$$(2) f(x) = \lim_{n \rightarrow \infty} \frac{|x|^n - 1}{|x|^n + 1} + x$$

[Sol] (i) When $|x| < 1$,

$$f(x) = x - 1$$

(ii) When $x = 1$,

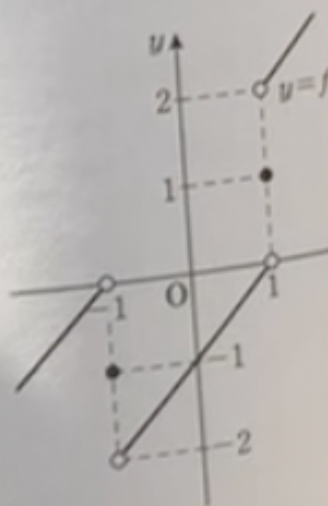
$$f(1) = 1$$

(iii) When $x = -1$,

$$f(-1) = -1$$

(iv) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{|x|^n}}{1 + \frac{1}{|x|^n}} + x = x + 1$$



From (i) ~ (iv), the graph of $y = f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x = \pm 1$.

Continuous and Discontinuous Functions

Name _____

Date / /

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Graph each given function and find x at which $f(x)$ is discontinuous.

$$(1) \quad f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n+1} + 1}{x^{2n} + 1}$$

[Sol] (i) When $|x| < 1$,

$$f(x) = 1$$

(ii) When $x = 1$,

$$f(1) = 1$$



$$\lim_{n \rightarrow \infty} x^{2n} = \lim_{n \rightarrow \infty} x^{2n+1} = 1$$

(iii) When $x = -1$,

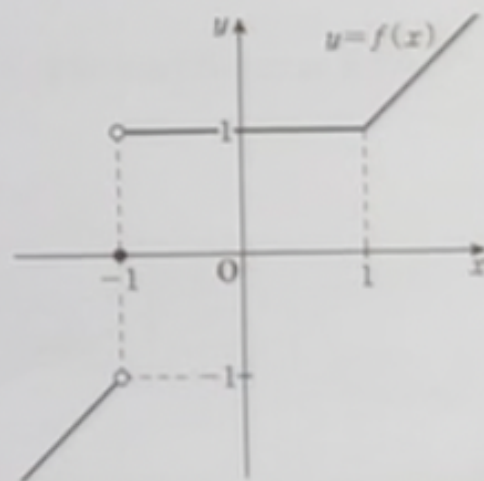
$$f(-1) = 0$$



$$\lim_{n \rightarrow \infty} x^{2n} = 1, \quad \lim_{n \rightarrow \infty} x^{2n+1} = -1$$

(iv) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x + \frac{1}{x^{2n}}}{1 + \frac{1}{x^{2n}}} = x$$

From (i) ~ (iv), the graph of $y = f(x)$ is as shown in the diagram.Also, $f(x)$ is discontinuous at $x = -1$.

N135b

$$(2) f(x) = \lim_{n \rightarrow \infty} (x + \sin^{2n} x) \quad (-\pi < x < \pi)$$

[Sol] (i) When $|\sin x| < 1$,

$$\text{i.e. } -\pi < x < -\frac{\pi}{2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \frac{\pi}{2} < x < \pi,$$

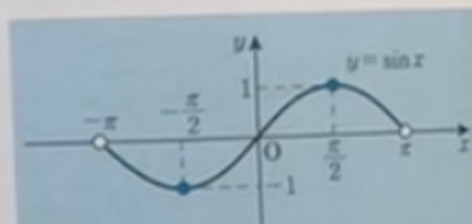
$$f(x) = x \quad \leftarrow \lim_{n \rightarrow \infty} \sin^{2n} x = 0$$

(ii) When $\sin x = 1$, i.e. $x = \frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1 \quad \leftarrow \lim_{n \rightarrow \infty} \sin^{2n} x = 1$$

(iii) When $\sin x = -1$, i.e. $x = -\frac{\pi}{2}$,

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 1 \quad \leftarrow \lim_{n \rightarrow \infty} \sin^{2n} x = 1$$



When

$$-\pi < x < -\frac{\pi}{2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \frac{\pi}{2} < x < \pi,$$

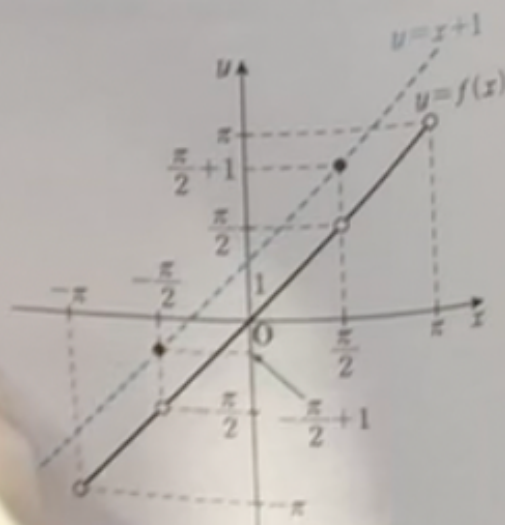
$$|\sin x| < 1$$

$$\text{When } x = \frac{\pi}{2}, \sin x = 1$$

$$\text{When } x = -\frac{\pi}{2}, \sin x = -1$$

From (i) ~ (iii), the graph of $y = f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x = \pm \frac{\pi}{2}$.



Continuous and Discontinuous Functions

Name _____

Date / /

Time : to :

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1. Find the constant a for which each given function $f(x)$ is continuous for all values of x .

Ex.

$$f(x) = \begin{cases} -ax^2 - 4a - 1 & (x \leq -2) \\ -ax - 9 & (x > -2) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = -2$. $\leftarrow f(-2) = \lim_{x \rightarrow -2} f(x)$ needs to be satisfied.

$$f(-2) = -a \cdot (-2)^2 - 4a - 1 = -8a - 1$$

$$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} (-ax - 9) = 2a - 9 \quad \leftarrow$$

$$\therefore -8a - 1 = 2a - 9$$

$$\therefore a = \frac{4}{5}$$

As $x \rightarrow -2^+$,

$$f(x) = -ax - 9$$

$$(1) \quad f(x) = \begin{cases} x^2 & (x \leq 2) \\ ax + 1 & (x > 2) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = 2$.

$$f(2) = 2^2 = 4$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (ax + 1) = 2a + 1$$

$$\therefore 4 = 2a + 1$$

$$\therefore a = \frac{3}{2}$$

$$(2) \quad f(x) = \begin{cases} |2x - 5| + 9a & (x < 3) \\ a^2x^2 + ax + 2 & (x \geq 3) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = 3$.

$$f(3) = a^2 \cdot 3^2 + a \cdot 3 + 2 = 9a^2 + 3a + 2$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (|2x - 5| + 9a) = 9a + 1$$

$$\therefore 9a^2 + 3a + 2 = 9a + 1$$

$$9a^2 - 6a + 1 = 0$$

$$(3a - 1)^2 = 0$$

$$\therefore a = \frac{1}{3}$$

2. Find the constants a and b for which each given function $f(x)$ is continuous for all values of x .

$$(1) \quad f(x) = \begin{cases} \frac{2}{x} & (x < -1, 2 < x) \\ ax+b & (-1 \leq x \leq 2) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = -1, 2$.

$$f(-1) = a \cdot (-1) + b = -a + b, \quad f(2) = a \cdot 2 + b = 2a + b$$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{2}{x} = -2, \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{2}{x} = 1$$

$$\therefore \begin{cases} -a + b = -2 & \dots \textcircled{1} \\ 2a + b = 1 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 1$, $b = -1$

$$(2) \quad f(x) = \begin{cases} 1 & (x \leq 0) \\ \frac{ax^2}{1 - \cos x} & (0 < x < \pi) \\ b & (x \geq \pi) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = 0, \pi$.

$$f(0) = 1, \quad f(\pi) = b$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{ax^2}{1 - \cos x} = \lim_{x \rightarrow 0^+} \frac{ax^2(1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \quad \leftarrow \text{N124} \\ &= \lim_{x \rightarrow 0^+} \frac{ax^2(1 + \cos x)}{1 - \cos^2 x} = \lim_{x \rightarrow 0^+} \frac{ax^2(1 + \cos x)}{\sin^2 x} \\ &= \lim_{x \rightarrow 0^+} \frac{a(1 + \cos x)}{\left(\frac{\sin x}{x}\right)^2} = 2a \quad \leftarrow \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \end{aligned}$$

$$\lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^-} \frac{ax^2}{1 - \cos x} = \frac{1}{2} \pi^2 a$$

$$\therefore \begin{cases} 1 = 2a & \dots \textcircled{1} \\ b = \frac{1}{2} \pi^2 a & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = \frac{1}{2}$, $b = \frac{\pi^2}{4}$

Continuous and Discontinuous Functions

Name _____

Date / /

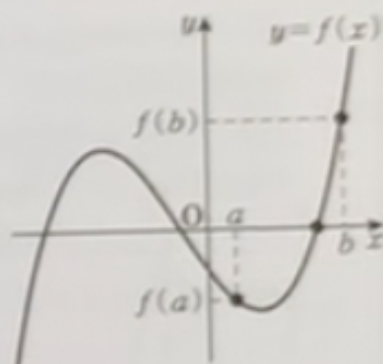
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Given that the function $f(x)$ is continuous on the closed interval $[a, b]$, then its graph has no break between points $(a, f(a))$ and $(b, f(b))$.

If $f(a)$ and $f(b)$ have different signs, then the graph intersects with the x -axis between a and b .

Since the x -coordinates of these points are solutions for the equation $f(x)=0$, the following statement is true.



Intermediate Value Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and $f(a) \cdot f(b) < 0$, then the equation $f(x)=0$ has at least one real solution in the interval $a < x < b$.

Ex. Prove that equation $x^3 + x^2 - 2x - 3 = 0$ has at least one real solution in the interval $0 < x < 2$.

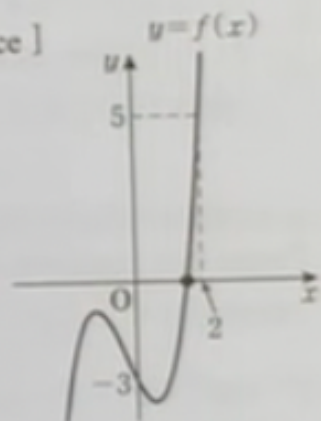
[Reference]

[Sol] Let $f(x) = x^3 + x^2 - 2x - 3$.

The function $f(x)$ is continuous on the interval $[0, 2]$.

Also, $f(0) = -3 < 0$, $f(2) = 5 > 0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $0 < x < 2$.

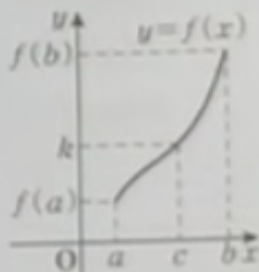


Answers: -3, 5

The interval $a \leq x \leq b$ is called a **closed interval** and the interval $a < x < b$ is called an **open interval**. They are expressed as $[a, b]$ and (a, b) , respectively.

Generally, the Intermediate Value Theorem is explained as follows:

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, then there is at least one value c that satisfies $f(c)=k$ and $a < c < b$ for any value k which lies between $f(a)$ and $f(b)$.



1. Prove that equation $x^4 - 5x + 2 = 0$ has at least one real solution in the interval $0 < x < 1$.

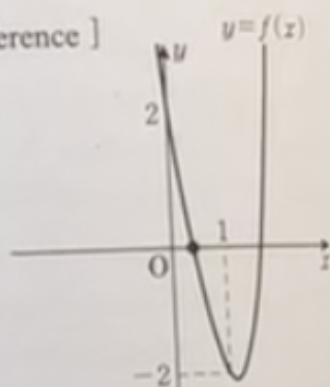
[Sol] Let $f(x) = x^4 - 5x + 2$.

The function $f(x)$ is continuous on the interval $[0, 1]$.

Also, $f(0) = 2 > 0$, $f(1) = -2 < 0$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has at least one real solution in the interval $0 < x < 1$.

[Reference]



2. Prove that equation $\log_2 x + x - 2 = 0$ has at least one real solution in the interval $1 < x < 2$.

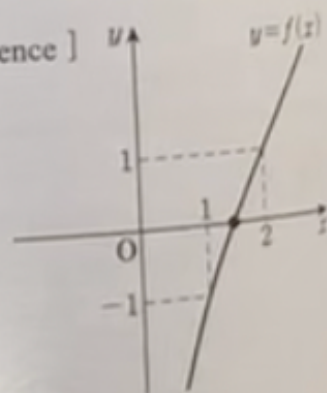
[Sol] Let $f(x) = \log_2 x + x - 2$.

The function $f(x)$ is continuous on the interval $[1, 2]$.

Also, $f(1) = -1 < 0$, $f(2) = 1 > 0$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has at least one real solution in the interval $1 < x < 2$.

[Reference]



3. Prove that equation $x = \cos x$ has at least one real solution in the interval $0 < x < \pi$.

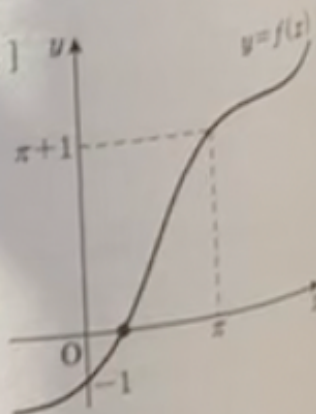
[Sol] Let $f(x) = x - \cos x$.

The function $f(x)$ is continuous on the interval $[0, \pi]$.

Also, $f(0) = -1 < 0$, $f(\pi) = \pi + 1 > 0$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has at least one real solution in the interval $0 < x < \pi$.

[Reference]



Continuous and Discontinuous Functions

Name _____

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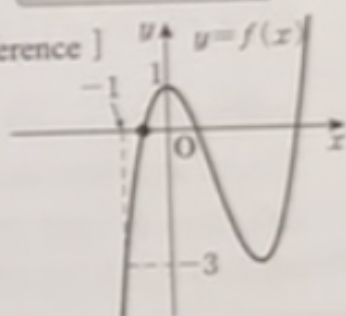
Ex. Prove that equation $x^3 - 3x^2 + 1 = 0$ has at least one negative solution.

[Sol] Let $f(x) = x^3 - 3x^2 + 1$. The function $f(x)$ is continuous for all real numbers x .

Also, $f(0) = 1 > 0$, $f(-1) = -3 < 0$ [Reference]

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has real solutions in the interval $-1 < x < 0$; that is, at least one negative solution.

Finding negative number x satisfying $f(0) \cdot f(x) < 0$



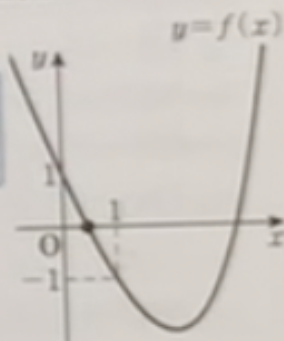
1. Prove that equation $2^x - 3x = 0$ has at least one positive solution.

[Sol] Let $f(x) = 2^x - 3x$. The function $f(x)$ is continuous for all real numbers x .

Also, $f(0) = 1 > 0$, $f(1) = -1 < 0$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has real solutions in the interval $0 < x < 1$; that is, at least one positive solution.

[Reference]
Finding positive number x satisfying $f(0) \cdot f(x) < 0$



2. Prove that equation $x - 2\sin x = k$ ($k > 0$) has at least one positive solution.

$$\left(\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0 \right)$$

[Sol] Let $f(x) = x - 2\sin x - k$ ($k > 0$). The function $f(x)$ is continuous for all real numbers x .

Also, $f(0) = -k < 0$ and

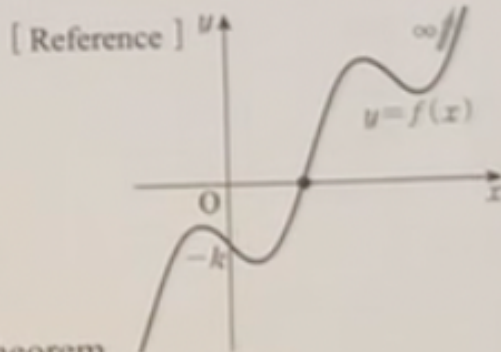
$$\text{since } f(x) = x \left(1 - \frac{2\sin x}{x} - \frac{k}{x} \right),$$

$$\lim_{x \rightarrow \infty} f(x) = \infty > 0$$

Therefore, from the Intermediate Value Theorem,

$f(x) = 0$ has real solutions in the interval

$0 < x < \infty$; that is, at least one positive solution.

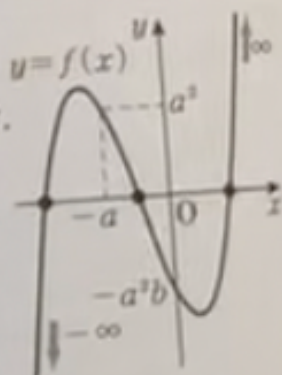


3. Given $a > 0$ and $b > 0$, prove that cubic equation $(x+a)^2(x-b)+x^2=0$ has one positive and two negative solutions.

(Use the fact that a cubic equation cannot have more than three solutions.)

[Sol] Let $f(x) = (x+a)^2(x-b) + x^2$.

[Reference]



The function $f(x)$ is continuous for all real numbers x .

Also, $f(0) = -a^2b < 0$, $f(-a) = a^3 > 0$ and

since $f(x) = x^3 \left[\left(1 + \frac{a}{x}\right)^2 \left(1 - \frac{b}{x}\right) + \frac{1}{x} \right]$,

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow \infty} f(x) = \infty$$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has real solutions in each interval $-\infty < x < -a$, $-a < x < 0$ and $0 < x < \infty$.

Since a cubic equation cannot have more than three solutions, $f(x) = 0$ has one positive and two negative solutions.

Alternative Solution

Let $f(x) = (x+a)^2(x-b) + x^2$.

The function $f(x)$ is continuous for all real numbers x .

Also $f(-a) = a^3 > 0$, $f(0) = -a^2b < 0$, $f(b) = b^3 > 0$ and

since $f(x) = x^3 \left[\left(1 + \frac{a}{x}\right)^2 \left(1 - \frac{b}{x}\right) + \frac{1}{x} \right]$,

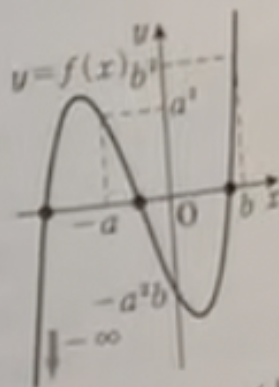
$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

Therefore, from the Intermediate Value Theorem,

$f(x) = 0$ has real solutions in each interval $-\infty < x < -a$, $-a < x < 0$ and $0 < x < b$.

Since a cubic equation cannot have more than three solutions, $f(x) = 0$ has one positive and two negative solutions.

[Reference]



Continuous and Discontinuous Functions

Name _____

Date / /

Time : to :

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1. Show that the following infinite series converges for all real numbers x . Then, let $f(x)$ be the sum. Graph the function $y=f(x)$ and find x at which $f(x)$ is discontinuous.

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^{n-1}} + \dots$$

[Sol] This infinite series is the infinite geometric series whose 1st term is x^2 and common ratio is $\frac{1}{1+x^2}$. ← N83

(i) When $x \neq 0$,

$$1+x^2 > 1; \text{ therefore, } \left| \frac{1}{1+x^2} \right| < 1$$

Thus, this infinite geometric series converges.

$$\text{The sum is } \frac{x^2}{1 - \frac{1}{1+x^2}} = 1+x^2.$$

(ii) When $x=0$, all terms become 0.

Therefore, the series converges and the sum is 0.

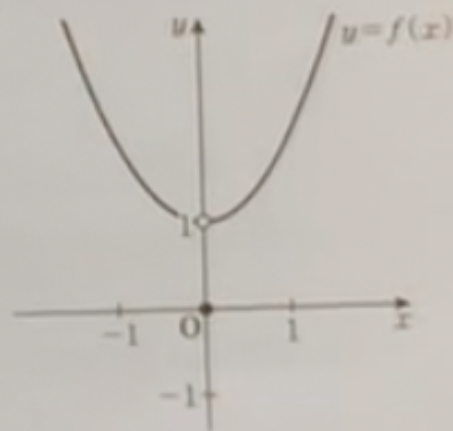
From (i) and (ii), this infinite series converges for all real numbers x .

Thus,

$$f(x) = \begin{cases} 1+x^2 & (x \neq 0) \\ 0 & (x=0) \end{cases}$$

Therefore, the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=0$.



2. Solve the following questions.

(1) Determine $f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} - x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$.

[Sol] (i) When $|x| < 1$,

$$f(x) = ax^2 + bx$$

$$\leftarrow \lim_{n \rightarrow \infty} x^{2n} = \lim_{n \rightarrow \infty} x^{2n-1} = 0$$

(ii) When $x = 1$,

$$f(1) = \frac{a+b}{2}$$

$$\leftarrow \lim_{n \rightarrow \infty} x^{2n} = \lim_{n \rightarrow \infty} x^{2n-1} = 1$$

(iii) When $x = -1$,

$$f(-1) = \frac{a-b+2}{2}$$

$$\leftarrow \begin{aligned} \lim_{n \rightarrow \infty} x^{2n} &= 1 \\ \lim_{n \rightarrow \infty} x^{2n-1} &= -1 \end{aligned}$$

(iv) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{x} + \frac{a}{x^{2n-2}} + \frac{b}{x^{2n-1}}}{1 + \frac{1}{x^{2n}}} = 1 - \frac{1}{x}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{x^{2n}} &= 0 \\ &= \lim_{n \rightarrow \infty} \frac{1}{x^{2n-1}} = 0 \\ &= \lim_{n \rightarrow \infty} \frac{1}{x^{2n-2}} = 0 \end{aligned}$$

(2) Find the constants a and b for which the function $f(x)$ determined above is continuous for all x .

[Sol] From (1), if $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = \pm 1$.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (ax^2 + bx) = a + b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \left(1 - \frac{1}{x}\right) = 0$$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \left(1 - \frac{1}{x}\right) = 2$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (ax^2 + bx) = a - b$$

$$\therefore \begin{cases} \frac{a+b}{2} = a+b=0 & \dots \textcircled{1} \\ \frac{a-b+2}{2} = 2 = a-b & \dots \textcircled{2} \end{cases} \leftarrow \begin{aligned} f(1) &= \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \\ f(-1) &= \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) \end{aligned}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 1$, $b = -1$

Continuous and Discontinuous Functions

Name _____

Date / /

Time : to :

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1. Examine if the following function is continuous at $x=1$.

➡ N131

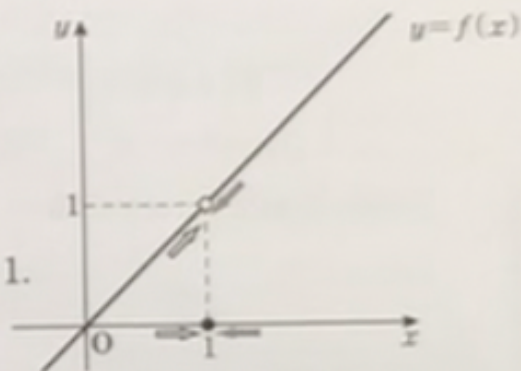
$$f(x) = \begin{cases} \frac{x^2-x}{x-1} & (x \neq 1) \\ 0 & (x = 1) \end{cases}$$

[Sol] $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x(x-1)}{x-1} = \lim_{x \rightarrow 1} x = 1$

Also, $f(1) = 0$

$\therefore \lim_{x \rightarrow 1} f(x) \neq f(1)$

Therefore, $f(x)$ is **discontinuous** at $x=1$.



2. Graph the following function and find x at which $f(x)$ is discontinuous.

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^2(1 - |x|^n)}{1 + |x|^n}$$

➡ N134

[Sol] (i) When $|x| < 1$,

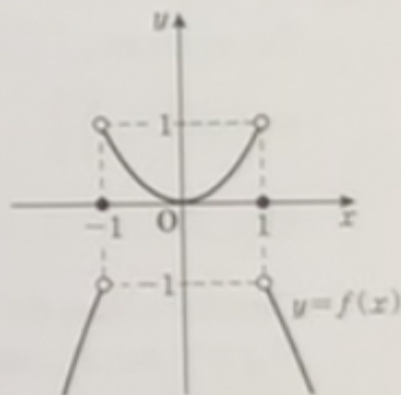
$$f(x) = x^2$$

(ii) When $x = \pm 1$,

$$f(x) = 0$$

(iii) When $|x| > 1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^2 \left(\frac{1}{|x|^n} - 1 \right)}{\frac{1}{|x|^n} + 1} = -x^2$$



From (i) ~ (iii), the graph of $y = f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x = \pm 1$.

3. Find the constants a and b for which the following function $f(x)$ is continuous for all x . ➡ N136

$$f(x) = \begin{cases} x - x^2 & (x < -2, 3 < x) \\ ax + b & (-2 \leq x \leq 3) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = -2, 3$.

$$f(-2) = a \cdot (-2) + b = -2a + b, \quad f(3) = a \cdot 3 + b = 3a + b$$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (x - x^2) = -6, \quad \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x - x^2) = -6$$

$$\therefore \begin{cases} -2a + b = -6 & \dots \textcircled{1} \\ 3a + b = -6 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 0$, $b = -6$

4. Prove that equation $\tan x = \cos x$ has at least one real solution in the interval $0 < x < \frac{\pi}{4}$. ➡ N137

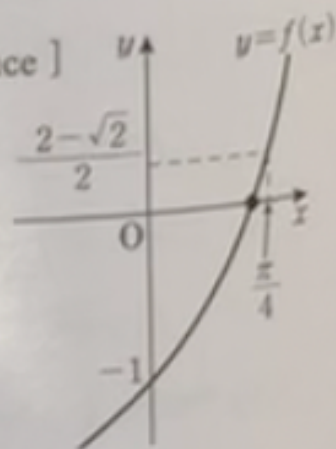
[Sol] Let $f(x) = \tan x - \cos x$.

The function $f(x)$ is continuous on the interval $\left[0, \frac{\pi}{4}\right]$.

$$\text{Also, } f(0) = -1 < 0, \quad f\left(\frac{\pi}{4}\right) = \frac{2 - \sqrt{2}}{2} > 0$$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has at least one real solution in the interval $0 < x < \frac{\pi}{4}$.

[Reference]



Differentiation 1

Name _____

Date / /

Time : to :

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Given the function $f(x)$, if the limit value $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists, then it is called the **derivative value** of $f(x)$ at $x=a$ and expressed as $f'(a)$.
In this case, $f(x)$ is said to be **differentiable** at $x=a$.

For the following functions, find the derivative values at x indicated in the brackets [].

Ex. $f(x) = x^3 - x$ [$x=1$]

[Sol] $f'(1) = \lim_{h \rightarrow 0} \frac{[(1+h)^3 - (1+h)] - (1^3 - 1)}{h}$ ←

$$= \lim_{h \rightarrow 0} \frac{h^3 + 3h^2 + 2h}{h}$$

$$= \lim_{h \rightarrow 0} (h^2 + 3h + 2)$$

$$= 2$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

(1) $f(x) = x^2 + x + 1$ [$x=2$]

[Sol] $f'(2) = \lim_{h \rightarrow 0} \frac{[(2+h)^2 + (2+h) + 1] - (2^2 + 2 + 1)}{h}$

$$= \lim_{h \rightarrow 0} \frac{h^2 + 5h}{h}$$

$$= \lim_{h \rightarrow 0} (h + 5)$$

$$= 5$$

[Reference] L44

Let $a+h=x$. Then, $h=x-a$. As $h \rightarrow 0$, $x \rightarrow a$. Therefore, $f'(a)$ can also be expressed as

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a}$$

N141b

$$(2) \quad f(x) = \frac{1}{x} \quad (x=3)$$

$$\begin{aligned} \text{[Sol]} \quad f'(3) &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{3h(3+h)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} \\ &= -\frac{1}{9} \end{aligned}$$

$$(3) \quad f(x) = \frac{1}{x^3} \quad (x=1)$$

$$\begin{aligned} \text{[Sol]} \quad f'(1) &= \lim_{h \rightarrow 0} \frac{\frac{1}{(1+h)^3} - \frac{1}{1^3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(h^3 + 3h^2 + 3h)}{h(1+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{-(h^2 + 3h + 3)}{(1+h)^3} \\ &= -3 \end{aligned}$$

$$(4) \quad f(x) = \sqrt{x} \quad (x=1)$$

$$\begin{aligned} \text{[Sol]} \quad f'(1) &= \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)} \end{aligned}$$

Multiplying the numerator and the denominator by $\sqrt{1+h} + 1$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h} + 1} \end{aligned}$$

Differentiation 1

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If the function $f(x)$ is differentiable for each value a on a certain interval, the function corresponding to the derivative value $f'(a)$ on this interval is called the **derivative** of $f(x)$ and expressed as $f'(x)$. The process of finding the derivative $f'(x)$ is called **differentiating** the function $f(x)$.

Using the definition of the derivative, differentiate the following functions.

Ex.

$$f(x) = \frac{1}{x^2}$$

$$\begin{aligned}
 \text{[Sol]} \quad f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} && \leftarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-(2hx + h^2)}{h(x+h)^2 x^2} \\
 &= \lim_{h \rightarrow 0} \frac{-(2x + h)}{(x+h)^2 x^2} \\
 &= -\frac{2}{x^3}
 \end{aligned}$$

$$(1) \quad f(x) = \frac{1}{x+1}$$

$$\begin{aligned}
 \text{[Sol]} \quad f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h(x+h+1)(x+1)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(x+h+1)(x+1)} \\
 &= -\frac{1}{(x+1)^2}
 \end{aligned}$$

The derivative of the function $y = f(x)$ is also expressed by symbols such as

y' , $[f(x)]'$, $\frac{dy}{dx}$ and $\frac{d}{dx}f(x)$. $\frac{dy}{dx}$ is read as "dy, dx."

N142b

$$(2) f(x) = \frac{x}{x-1}$$

$$\begin{aligned} \text{[Sol]} f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{(x+h)-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h(x+h-1)(x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} \\ &= -\frac{1}{(x-1)^2} \end{aligned}$$

$$(3) f(x) = \sqrt{x}$$

$$\begin{aligned} \text{[Sol]} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Multiplying the numerator
and the denominator by
 $\sqrt{x+h} + \sqrt{x}$

$$(4) f(x) = \sqrt{2x}$$

$$\begin{aligned} \text{[Sol]} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{[\sqrt{2(x+h)} - \sqrt{2x}][\sqrt{2(x+h)} + \sqrt{2x}]}{h[\sqrt{2(x+h)} + \sqrt{2x}]} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h[\sqrt{2(x+h)} + \sqrt{2x}]} \\ &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{2(x+h)} + \sqrt{2x}} \\ &= \frac{1}{\sqrt{2x}} \end{aligned}$$

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Properties of Derivatives

When k is a constant and n is a positive integer,

if $y = x^n$, then $y' = nx^{n-1}$

if $y = kf(x)$, then $y' = kf'(x)$

if $y = f(x) + g(x)$, then $y' = f'(x) + g'(x)$

if $y = f(x) - g(x)$, then $y' = f'(x) - g'(x)$

Differentiate the following functions.

Ex.

$y = x^3 + 3x^2 - 4x + 2$

[Sol] $y' = 3x^2 + 6x - 4$

(1) $y = 2x^3 + 5x$

[Sol] $y' = 6x^2 + 5$

(5) $y = 4x^3 + \frac{2}{3}x^2 - \frac{3}{5}x$

[Sol] $y' = 12x^2 + \frac{4}{3}x - \frac{3}{5}$

(2) $y = x^2 - 2x - 1$

[Sol] $y' = 2x - 2$

(6) $y = -x^3 + 5x^2 - 4x + 2$

[Sol] $y' = -3x^2 + 10x - 4$

(3) $y = x^4 - 2x^3 + 3x - 9$

[Sol] $y' = 4x^3 - 6x^2 + 3$

(7) $y = -2x^3 + \frac{1}{2}x^2 - 4x + 1$

[Sol] $y' = -6x^2 + x - 4$

(4) $y = \frac{2}{3}x^3 - x^2 + 2x - 3$

[Sol] $y' = 2x^2 - 2x + 2$

(8) $y = -\frac{3}{4}x^3 + \frac{5}{2}x^2 + x - \frac{1}{2}$

[Sol] $y' = -\frac{9}{4}x^2 + 5x + 1$

N143b

$$(9) \quad y = 2(3x - 2)$$

$$[\text{Sol}] \quad y = \boxed{6}x - \boxed{4}$$

$$y' = \boxed{6}$$

$$(10) \quad y = x(x + 1)$$

$$[\text{Sol}] \quad y = x^2 + x$$

$$y' = 2x + 1$$

$$(11) \quad y = \frac{2}{3}x^2(x - 3)$$

$$[\text{Sol}] \quad y = \frac{2}{3}x^3 - 2x^2$$

$$y' = 2x^2 - 4x$$

$$(12) \quad y = x^3(x^2 - 4x + 1)$$

$$[\text{Sol}] \quad y = x^5 - 4x^4 + x^3$$

$$y' = 5x^4 - 16x^3 + 3x^2$$

$$(13) \quad y = (x - 1)(x + 2)$$

$$[\text{Sol}] \quad y = x^2 + x - 2$$

$$y' = 2x + 1$$

$$(14) \quad y = (3x^2 - 1)(x + 2)$$

$$[\text{Sol}] \quad y = 3x^3 + 6x^2 - x - 2$$

$$y' = 9x^2 + 12x - 1$$

Differentiation 1

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Given the two functions $f(x)$ and $g(x)$, prove the following equality using the definition of the derivative.

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

[Sol] Let $y = f(x)g(x)$.

$$\begin{aligned} y' &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right] \\ &= f'(x)g(x) + f(x)g'(x) \quad \leftarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x) \end{aligned}$$

Product Rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Using the formula above, differentiate the following functions.

Ex. $y = (x^2 + 1)(x^2 - 2)$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^2 + 1)'(x^2 - 2) + (x^2 + 1)(x^2 - 2)' \\ &= 2x(x^2 - 2) + (x^2 + 1) \cdot 2x \\ &= 4x^3 - 2x \end{aligned}$$

(1) $y = (3x^2 + 1)(2x - 3)$

$$\begin{aligned} \text{[Sol]} \quad y' &= (3x^2 + 1)'(2x - 3) + (3x^2 + 1)(2x - 3)' \\ &= 6x(2x - 3) + (3x^2 + 1) \cdot 2 \\ &= 18x^2 - 18x + 2 \quad [= 2(9x^2 - 9x + 1)] \end{aligned}$$

N144b

$$(2) \quad y = (2x+4)(-x-3)$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (2x+4)'(-x-3) + (2x+4)(-x-3)' \\ &= 2(-x-3) + (2x+4) \cdot (-1) \\ &= -4x-10 \quad [= -2(2x+5)] \end{aligned}$$

$$(3) \quad y = (2x^2+3)(-x^2+5x)$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (2x^2+3)'(-x^2+5x) + (2x^2+3)(-x^2+5x)' \\ &= 4x(-x^2+5x) + (2x^2+3)(-2x+5) \\ &= -8x^3+30x^2-6x+15 \end{aligned}$$

$$(4) \quad y = (x^2-2x-3)(x+4)$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^2-2x-3)'(x+4) + (x^2-2x-3)(x+4)' \\ &= (2x-2)(x+4) + (x^2-2x-3) \cdot 1 \\ &= 3x^2+4x-11 \end{aligned}$$

$$(5) \quad y = (x^2-2)(x^3+3x-5)$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^2-2)'(x^3+3x-5) + (x^2-2)(x^3+3x-5)' \\ &= 2x(x^3+3x-5) + (x^2-2)(3x^2+3) \\ &= 5x^4+3x^2-10x-6 \end{aligned}$$

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1. Differentiate the following functions.

Ex. $y = x(x+1)(2x-3)$

[Sol] $y = (x^2+x)(2x-3)$

$$y' = (2x+1)(2x-3) + (x^2+x) \cdot 2$$

$$= 6x^2 - 2x - 3$$



$$x(x+1) = x^2 + x$$



$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

(1) $y = 2x(x-1)(3x-1)$

[Sol] $y = (2x^2-2x)(3x-1)$

$$y' = (4x-2)(3x-1) + (2x^2-2x) \cdot 3$$

$$= 18x^2 - 16x + 2 \quad [= 2(9x^2 - 8x + 1)]$$

(2) $y = (x+1)(x+2)(x+3)$

[Sol] $y = (x^2+3x+2)(x+3)$

$$y' = (2x+3)(x+3) + (x^2+3x+2) \cdot 1$$

$$= 3x^2 + 12x + 11$$

(3) $y = (x-2)(x+1)(2x+1)$

[Sol] $y = (x^2-x-2)(2x+1)$

$$y' = (2x-1)(2x+1) + (x^2-x-2) \cdot 2$$

$$= 6x^2 - 2x - 5$$

N145b

$$(4) \quad y = (3x+2)(x-1)(2x+5)$$

$$[\text{Sol}] \quad y = (3x^2 - x - 2)(2x+5)$$

$$\begin{aligned} y' &= (6x-1)(2x+5) + (3x^2-x-2) \cdot 2 \\ &= 18x^2 + 26x - 9 \end{aligned}$$

$$(5) \quad y = (x-3)(x+1)^2(x+3)$$

$$[\text{Sol}] \quad y = (x^2-9)(x^2+2x+1)$$

$$\begin{aligned} y' &= 2x(x^2+2x+1) + (x^2-9)(2x+2) \\ &= 4x^3 + 6x^2 - 16x - 18 \quad \left[= 2(x+1)(2x^2+x-9) \right] \end{aligned}$$

2. Show that the derivative of the function $y = f(x)g(x)h(x)$ is given by the following expression.

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$[\text{Sol}] \quad \text{Since } y = [f(x)g(x)]h(x),$$

$$\begin{aligned} y' &= [f(x)g(x)]'h(x) + [f(x)g(x)]h'(x) \\ &= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

3. Using the equality shown in question 2, differentiate the following function.

$$y = (x+1)(x-2)(x+3)$$

$$\begin{aligned} [\text{Sol}] \quad y' &= 1 \cdot (x-2)(x+3) + (x+1) \cdot 1 \cdot (x+3) + (x+1)(x-2) \cdot 1 \\ &= 3x^2 + 4x - 5 \end{aligned}$$

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Given the two functions $f(x)$ and $g(x)$, prove the following equalities using the definition of the derivative.

$$[1] \quad \left[\frac{1}{g(x)} \right]' = -\frac{g'(x)}{[g(x)]^2}$$

[Sol] Let $y = \frac{1}{g(x)}$.

$$y' = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{g(x+h)} - \frac{1}{g(x)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-\frac{g(x+h) - g(x)}{g(x+h)g(x)} \right]$$

$$= \lim_{h \rightarrow 0} \left[-\frac{1}{g(x+h)g(x)} \cdot \frac{g(x+h) - g(x)}{h} \right]$$

$$= -\frac{1}{[g(x)]^2} \cdot g'(x)$$

$$= -\frac{g'(x)}{[g(x)]^2}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$[2] \quad \left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad (\text{Use the equality in [1].})$$

[Sol] $\left[\frac{f(x)}{g(x)} \right]' = \left[f(x) \cdot \frac{1}{g(x)} \right]'$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left[\frac{1}{g(x)} \right]'$$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left[-\frac{g'(x)}{[g(x)]^2} \right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\begin{aligned} [f(x)g(x)]' &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

From [1]

Answers: $\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

From the results on side a, the following formulas are true.

Quotient Rule

$$\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad \left[\frac{1}{g(x)} \right]' = -\frac{g'(x)}{[g(x)]^2}$$

Using the formulas above, differentiate the following functions.

Ex.

$$y = \frac{x}{x^2+1}$$

$$[\text{Sol}] \quad y' = \frac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2} = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = -\frac{x^2-1}{(x^2+1)^2}$$

$$(1) \quad y = \frac{x}{3x-1}$$

$$[\text{Sol}] \quad y' = \frac{(x)'(3x-1) - x(3x-1)'}{(3x-1)^2} = \frac{1 \cdot (3x-1) - x \cdot 3}{(3x-1)^2} = -\frac{1}{(3x-1)^2}$$

$$(2) \quad y = \frac{x-1}{x+1}$$

$$[\text{Sol}] \quad y' = \frac{(x-1)'(x+1) - (x-1)(x+1)'}{(x+1)^2} = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$(3) \quad y = \frac{1}{4x-1}$$

$$[\text{Sol}] \quad y' = -\frac{(4x-1)'}{(4x-1)^2} = -\frac{4}{(4x-1)^2}$$

$$(4) \quad y = \frac{3}{x^2+1}$$

$$[\text{Sol}] \quad y' = -\frac{3(x^2+1)'}{(x^2+1)^2} = -\frac{3 \cdot 2x}{(x^2+1)^2} = -\frac{6x}{(x^2+1)^2}$$

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Differentiate the following functions.

$$(1) \quad y = \frac{x^2}{2x-1}$$

$$[\text{Sol}] \quad y' = \frac{2x(2x-1) - x^2 \cdot 2}{(2x-1)^2} \quad \leftarrow \quad \left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{2x^2 - 2x}{(2x-1)^2} \quad \left[= \frac{2x(x-1)}{(2x-1)^2} \right]$$

$$(2) \quad y = \frac{x^2-1}{x^2+1}$$

$$[\text{Sol}] \quad y' = \frac{2x(x^2+1) - (x^2-1) \cdot 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

$$(3) \quad y = \frac{1}{x^2+2x+3}$$

$$[\text{Sol}] \quad y' = -\frac{2x+2}{(x^2+2x+3)^2} \quad \leftarrow \quad \left[\frac{1}{g(x)} \right]' = -\frac{g'(x)}{[g(x)]^2}$$

$$\left[= -\frac{2(x+1)}{(x^2+2x+3)^2} \right]$$

$$(4) \quad y = \frac{x}{x^2-x+1}$$

$$[\text{Sol}] \quad y' = \frac{1 \cdot (x^2-x+1) - x(2x-1)}{(x^2-x+1)^2}$$

$$= -\frac{x^2-1}{(x^2-x+1)^2} \quad \left[= -\frac{(x+1)(x-1)}{(x^2-x+1)^2} \right]$$

N147b

$$(5) \quad y = \frac{2x^2 - 3x + 5}{x^2 + 2}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(4x-3)(x^2+2) - (2x^2-3x+5) \cdot 2x}{(x^2+2)^2} \\ &= \frac{3x^2 - 2x - 6}{(x^2+2)^2} \end{aligned}$$

$$(6) \quad y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(2x-1)(x^2+x+1) - (x^2-x+1)(2x+1)}{(x^2+x+1)^2} \\ &= \frac{2x^2-2}{(x^2+x+1)^2} \left[= \frac{2(x+1)(x-1)}{(x^2+x+1)^2} \right] \end{aligned}$$

$$(7) \quad y = \frac{(x^2+1)(x-1)}{x^3}$$

$$\begin{aligned} [\text{Sol}] \quad [(x^2+1)(x-1)]' &= 2x(x-1) + (x^2+1) \cdot 1 = 3x^2 - 2x + 1 \\ y' &= \frac{(3x^2 - 2x + 1) \cdot x^3 - (x^2+1)(x-1) \cdot 3x^2}{x^6} \\ &= \frac{x^3 - 2x + 3}{x^4} \end{aligned}$$

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Prove $(x^n)' = nx^{n-1}$ is true not only when n is a positive integer but also when n is a negative integer or 0.

[Sol] When n is a negative integer, let $n = -m$. Then, m is a positive integer.

$$(x^n)' = (x^{-m})' = \left(\frac{1}{x^m}\right)' = -\frac{(x^m)'}{(x^m)^2} = -\frac{mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = -mx^{m-1-m} = nx^{n-1}$$

Therefore, $(x^n)' = nx^{n-1}$ is also true when n is a negative integer.

Also, since $x^n = 1$ when $n = 0$, $(1)' = 0$; therefore, $(x^n)' = nx^{n-1}$ is true.

Answers: m-1, -m

Derivative of x^n

When n is an integer, $(x^n)' = nx^{n-1}$

Differentiate the following functions.

Ex.

$$y = \frac{1}{x^3}$$

[Sol] $y = x^{-3}$

$$y' = -3x^{-4} = -\frac{3}{x^4}$$

$$(2) \quad y = \frac{1}{3x^3}$$

[Sol] $y = \frac{1}{3}x^{-3}$

$$y' = \frac{1}{3} \cdot (-3)x^{-4} = -\frac{1}{x^4}$$

$$(1) \quad y = \frac{1}{x^4}$$

[Sol] $y = x^{-4}$

$$y' = -4x^{-5} = -\frac{4}{x^5}$$

$$(3) \quad y = x + \frac{2}{x}$$

[Sol] $y = x + 2x^{-1}$

$$y' = 1 + 2 \cdot (-1)x^{-2} = 1 - \frac{2}{x^2}$$

In the case of Ex., $y' = -3x^{-4}$ is not a wrong answer. However, generally, the answer has to be written in the same form as the question. (For Ex., use the fraction form such as $\frac{1}{x^3}$.)

N148b

$$(4) \quad y = \frac{x^2+3}{x}$$

$$[\text{Sol}] \quad y = \boxed{x} + \frac{3}{x} = \boxed{x} + 3x^{\boxed{-1}}$$

$$y' = 1 + 3 \cdot (-1)x^{-2} = 1 - \frac{3}{x^2} \quad \left[= \frac{x^2-3}{x^2} \right]$$

$$(5) \quad y = \frac{2x^2-x+5}{3x^3}$$

$$[\text{Sol}] \quad y = \frac{2}{3x} - \frac{1}{3x^2} + \frac{5}{3x^3} = \frac{2}{3}x^{-1} - \frac{1}{3}x^{-2} + \frac{5}{3}x^{-3}$$

$$y' = \frac{2}{3} \cdot (-1)x^{-2} - \frac{1}{3} \cdot (-2)x^{-3} + \frac{5}{3} \cdot (-3)x^{-4} = -\frac{2}{3x^2} + \frac{2}{3x^3} - \frac{5}{x^4}$$

$$\left[= -\frac{2x^2-2x+15}{3x^4} \right]$$

$$(6) \quad y = \frac{5x^3+3x^2-4}{2x^2}$$

$$[\text{Sol}] \quad y = \frac{5}{2}x + \frac{3}{2} - \frac{2}{x^2} = \frac{5}{2}x + \frac{3}{2} - 2x^{-2}$$

$$y' = \frac{5}{2} - 2 \cdot (-2)x^{-3} = \frac{5}{2} + \frac{4}{x^3} \quad \left[= \frac{5x^3+8}{2x^3} \right]$$

$$(7) \quad y = \frac{(3x-1)(2x+3)}{x^3}$$

$$[\text{Sol}] \quad y = \frac{6x^2+7x-3}{x^3} = \frac{6}{x} + \frac{7}{x^2} - \frac{3}{x^3} = 6x^{-1} + 7x^{-2} - 3x^{-3}$$

$$y' = 6 \cdot (-1)x^{-2} + 7 \cdot (-2)x^{-3} - 3 \cdot (-3)x^{-4} = -\frac{6}{x^2} - \frac{14}{x^3} + \frac{9}{x^4}$$

$$\left[= -\frac{6x^2+14x-9}{x^4} \right]$$

Differentiation 1

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

1. Given that $f(x) = \frac{ax+b}{x^2-x+1}$ satisfies $f(2) = \frac{4}{3}$ and $f'(0) = 2$, find the values of constants a and b .

$$[\text{Sol}] \quad f(2) = \frac{a \cdot 2 + b}{2^2 - 2 + 1} = \frac{2a + b}{3}$$

$$\text{Since } f(2) = \frac{4}{3}, \quad \frac{2a + b}{3} = \frac{4}{3}$$

$$\therefore 2a + b = 4 \quad \cdots \textcircled{1}$$

$$\begin{aligned} f'(x) &= \frac{a(x^2 - x + 1) - (ax + b)(2x - 1)}{(x^2 - x + 1)^2} \\ &= -\frac{ax^2 + 2bx - (a + b)}{(x^2 - x + 1)^2} \end{aligned}$$

$$\text{Since } f'(0) = 2, \quad a + b = 2 \quad \cdots \textcircled{2}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, \quad a = 2, \quad b = 0$$

N149b

2. For all natural numbers n , when $x \neq 1$, the following equality is true.

$$1 + x + x^2 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1} \quad \dots \textcircled{1}$$

Using this equality, find the following sums.

(1) $1 + 2x + 3x^2 + \dots + nx^{n-1}$

[Sol] Differentiating both sides of $\textcircled{1}$ with respect to x ,

$$\begin{aligned} & 1 + 2x + 3x^2 + \dots + nx^{n-1} \\ &= \frac{(n+1)x^n \cdot (x-1) - (x^{n+1} - 1) \cdot 1}{(x-1)^2} \\ &= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \end{aligned}$$

(2) $1 + \frac{2}{2} + \frac{3}{2^2} + \dots + \frac{n}{2^{n-1}}$

[Sol] From (1),

$$1 + 2x + 3x^2 + \dots + nx^{n-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \quad \dots \textcircled{2}$$

Substituting $x = \frac{1}{2}$ into $\textcircled{2}$,

$$\begin{aligned} & 1 + \frac{2}{2} + \frac{3}{2^2} + \dots + \frac{n}{2^{n-1}} \\ &= \frac{n \cdot \frac{1}{2^{n+1}} - (n+1) \cdot \frac{1}{2^n} + 1}{\left(\frac{1}{2} - 1\right)^2} \\ &= \frac{2^{n+1} - n - 2}{2^{n-1}} \left[= 4 - (n+2) \left(\frac{1}{2}\right)^{n-1} \right] \end{aligned}$$

Differentiation 1

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

1. Using the definition of the derivative, differentiate $f(x) = \frac{\sqrt{x}}{2}$. ➡ N142

$$\begin{aligned}
 \text{[Sol]} \quad f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{x+h}}{2} - \frac{\sqrt{x}}{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{2h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{2h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{h}{2h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{2(\sqrt{x+h} + \sqrt{x})} \\
 &= \frac{1}{4\sqrt{x}}
 \end{aligned}$$

2. Differentiate the following functions.

(1) $y = (x^2 + x - 3)(2x^2 - 5)$ ➡ N144

$$\begin{aligned}
 \text{[Sol]} \quad y' &= (2x+1)(2x^2-5) + (x^2+x-3) \cdot 4x \\
 &= 8x^3 + 6x^2 - 22x - 5
 \end{aligned}$$

(2) $y = (x+3)^2(x-1)$ ➡ N145

$$\begin{aligned}
 \text{[Sol]} \quad y &= (x^2 + 6x + 9)(x-1) \\
 y' &= (2x+6)(x-1) + (x^2+6x+9) \cdot 1 \\
 &= 3x^2 + 10x + 3 \quad [= (x+3)(3x+1)]
 \end{aligned}$$

Alternative Solution

$$y' = 1 \cdot (x+3)(x-1) + (x+3) \cdot 1 \cdot (x-1) + (x+3)^2 \cdot 1 = 3x^2 + 10x + 3 \quad [= (x+3)(3x+1)]$$

N150b

$$(3) \quad y = \frac{1}{x^2 - 5x + 3}$$

➡ N146

$$[\text{Sol}] \quad y' = -\frac{2x-5}{(x^2-5x+3)^2}$$

$$(4) \quad y = \frac{2x+1}{x^2+3x-1}$$

➡ N146

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{2(x^2+3x-1) - (2x+1)(2x+3)}{(x^2+3x-1)^2} \\ &= -\frac{2x^2+2x+5}{(x^2+3x-1)^2} \end{aligned}$$

$$(5) \quad y = x^2 - 2x + \frac{3}{x^2}$$

➡ N148

$$[\text{Sol}] \quad y = x^2 - 2x + 3x^{-2}$$

$$y' = 2x - 2 + 3 \cdot (-2)x^{-3} = 2x - 2 - \frac{6}{x^3}$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ y' = 2x - 2 - \frac{3 \cdot 2x}{x^4} = 2x - 2 - \frac{6}{x^3} \end{array} \right]$$

$$(6) \quad y = \frac{3x^2 - 2x + 5}{x}$$

➡ N148

$$[\text{Sol}] \quad y = 3x - 2 + \frac{5}{x} = 3x - 2 + 5x^{-1}$$

$$y' = 3 + 5 \cdot (-1)x^{-2} = 3 - \frac{5}{x^2} \quad \left[= \frac{3x^2 - 5}{x^2} \right]$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ y' = \frac{(6x-2) \cdot x - (3x^2-2x+5) \cdot 1}{x^2} = \frac{3x^2-5}{x^2} \end{array} \right]$$

Differentiation 2

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Given the two functions $f(x)$ and $g(x)$ where the range of $f(x)$ is within the domain of $g(x)$, let $u = f(x)$ and $y = g(u)$. Then, $y = g(u) = g(f(x))$ can be derived.

The function $g(f(x))$ is called the *composite function* of $f(x)$ and $g(x)$.

1. For each given pair of functions, find the composite functions $g(f(x))$ and $f(g(x))$.

Ex. $f(x) = x^2 - 1$, $g(x) = 2x + 1$

$$\begin{aligned} \text{[Sol]} \quad g(f(x)) &= g(x^2 - 1) \\ &= 2(x^2 - 1) + 1 \\ &= 2x^2 - 1 \end{aligned}$$

$$\begin{aligned} f(g(x)) &= f(2x + 1) \\ &= (2x + 1)^2 - 1 \\ &= 4x^2 + 4x \end{aligned}$$

(1) $f(x) = x^2 + 1$, $g(x) = 3x - 2$

$$\begin{aligned} \text{[Sol]} \quad g(f(x)) &= g(x^2 + 1) \\ &= 3(x^2 + 1) - 2 \\ &= 3x^2 + 1 \end{aligned}$$

$$\begin{aligned} f(g(x)) &= f(3x - 2) \\ &= (3x - 2)^2 + 1 \\ &= 9x^2 - 12x + 5 \end{aligned}$$

(2) $f(x) = x^2$, $g(x) = \sqrt{x+1}$

$$\begin{aligned} \text{[Sol]} \quad g(f(x)) &= g(x^2) \\ &= \sqrt{x^2 + 1} \end{aligned}$$

$$\begin{aligned} f(g(x)) &= f(\sqrt{x+1}) \\ &= (\sqrt{x+1})^2 \\ &= x + 1 \end{aligned}$$

Given that the composite function $y = f(g(x))$ can be differentiated when both $y = f(u)$ and $u = g(x)$ are differentiable, prove that the following equation is true using the definition of the derivative.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \leftarrow \text{For } \frac{d\bullet}{d\blacktriangle}, \text{ differentiating } \bullet \text{ with respect to } \blacktriangle$$

$$\begin{aligned} [\text{Sol}] \quad \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \right] \end{aligned}$$

$$\text{Let } g(x+h) - g(x) = k. \quad g(x+h) = g(x) + \boxed{k} = u + \boxed{k}$$

$$\text{Also, as } h \rightarrow 0, k \rightarrow \boxed{0}$$

$$\therefore \frac{dy}{dx} = \lim_{h \rightarrow 0} \left[\frac{f(u + \boxed{k}) - f(u)}{k} \cdot \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{k \rightarrow \boxed{0}} \frac{f(u + \boxed{k}) - f(u)}{k} \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(u) \cdot g'(x)$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Answers: $k, k, 0, k, 0, k$

Chain Rule I

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2. Using the formula above, differentiate $y = (x^2 + 1)^3$.

$$[\text{Sol}] \text{ Let } u = x^2 + 1, \quad y = u^3$$

$$\frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = \boxed{2x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \boxed{2x} = \boxed{6x(x^2 + 1)^2}$$

Differentiation 2

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For the formula on N151b, since $\frac{dy}{du} = f'(u)$ and $\frac{du}{dx} = g'(x)$ when $y = f(u)$ and $u = g(x)$, it can also be expressed as follows.

Chain Rule II

$$[f(g(x))]' = f'(g(x))g'(x)$$

Differentiate the following functions.

Ex.

$$y = (x^2 + 1)^4$$

←

$$y = \square^4$$

$$[\text{Sol}] \quad y' = 4(x^2 + 1)^3 \cdot (x^2 + 1)'$$

←

$$y' = 4 \square^3 \cdot \square'$$

$$= 4(x^2 + 1)^3 \cdot 2x$$

$$= 8x(x^2 + 1)^3$$

$$(1) \quad y = (x^3 + 2)^5$$

$$(4) \quad y = (1 - 2x^3)^{10}$$

$$[\text{Sol}] \quad y' = 5(x^3 + 2)^4 \cdot (x^3 + 2)'$$

$$[\text{Sol}] \quad y' = 10(1 - 2x^3)^9 \cdot (1 - 2x^3)'$$

$$= 5(x^3 + 2)^4 \cdot 3x^2$$

$$= 10(1 - 2x^3)^9 \cdot (-6x^2)$$

$$= 15x^2(x^3 + 2)^4$$

$$= -60x^2(1 - 2x^3)^9$$

$$(2) \quad y = (2x^2 + 5)^6$$

$$(5) \quad y = (3x^3 + x - 1)^2$$

$$[\text{Sol}] \quad y' = 6(2x^2 + 5)^5 \cdot (2x^2 + 5)'$$

$$[\text{Sol}] \quad y' = 2(3x^3 + x - 1) \cdot (3x^3 + x - 1)'$$

$$= 6(2x^2 + 5)^5 \cdot 4x$$

$$= 2(3x^3 + x - 1)(9x^2 + 1)$$

$$= 24x(2x^2 + 5)^5$$

$$(3) \quad y = (1 - x^2)^3$$

$$(6) \quad y = (x^3 - 7x + 1)^4$$

$$[\text{Sol}] \quad y' = 3(1 - x^2)^2 \cdot (1 - x^2)'$$

$$[\text{Sol}] \quad y' = 4(x^3 - 7x + 1)^3 \cdot (x^3 - 7x + 1)'$$

$$= 3(1 - x^2)^2 \cdot (-2x)$$

$$= 4(x^3 - 7x + 1)^3(3x^2 - 7)$$

$$= -6x(1 - x^2)^2$$

$$[-6x(1 + x)^2(1 - x)^2]$$

N152b

$$(7) \quad y = (3x^2 - 5)^{-4}$$

$$[\text{Sol}] \quad y' = -4(3x^2 - 5)^{-5} \cdot (3x^2 - 5)'$$

$$= -4(3x^2 - 5)^{-5} \cdot 6x$$

$$= -24x(3x^2 - 5)^{-5} \quad \left[= -\frac{24x}{(3x^2 - 5)^5} \right]$$

$$(8) \quad y = (2x^3 - x + 4)^{-3}$$

$$[\text{Sol}] \quad y' = -3(2x^3 - x + 4)^{-4} \cdot (2x^3 - x + 4)'$$

$$= -3(2x^3 - x + 4)^{-4} (6x^2 - 1) \quad \left[= -\frac{3(6x^2 - 1)}{(2x^3 - x + 4)^4} \right]$$

$$(9) \quad y = \frac{1}{(x^2 + 1)^3}$$

$$[\text{Sol}] \quad y = (x^2 + 1)^{-3}$$

$$y' = -3(x^2 + 1)^{-4} \cdot (x^2 + 1)'$$

$$= -3(x^2 + 1)^{-4} \cdot 2x$$

$$= -\frac{6x}{(x^2 + 1)^4} \quad \left[= -6x(x^2 + 1)^{-4} \right]$$

$$(10) \quad y = \frac{1}{(1 - 2x^3)^2}$$

$$[\text{Sol}] \quad y = (1 - 2x^3)^{-2}$$

$$y' = -2(1 - 2x^3)^{-3} \cdot (1 - 2x^3)'$$

$$= -2(1 - 2x^3)^{-3} \cdot (-6x^2)$$

$$= \frac{12x^2}{(1 - 2x^3)^3} \quad \left[= 12x^2(1 - 2x^3)^{-3} \right]$$

Generally, the answer has to be written in the same form as the question. (For (7), use the exponent form, such as $(3x^2 - 5)^{-4}$. For (9), use the fraction form, such as $\frac{1}{(x^2 + 1)^3}$.)

Differentiation 2

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

Differentiate the following functions.

Ex. $y = x(4x-1)^3$

$$[(4x-1)^3]' = 3(4x-1)^2 \cdot (4x-1)'$$

[Sol] $y' = 1 \cdot (4x-1)^3 + x \cdot 3(4x-1)^2 \cdot 4$

$$= (4x-1)^2 [(4x-1) + 12x]$$

$$= (4x-1)^2 (16x-1)$$

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

(1) $y = (3x-1)^2(x+1)$

[Sol] $y' = 2(3x-1) \cdot 3 \cdot (x+1) + (3x-1)^2 \cdot 1$

$$= (3x-1)[6(x+1) + (3x-1)]$$

$$= (3x-1)(9x+5)$$

(2) $y = (2x-1)^2(x+3)^3$

[Sol] $y' = 2(2x-1) \cdot 2 \cdot (x+3)^3 + (2x-1)^2 \cdot 3(x+3)^2 \cdot 1$

$$= (2x-1)(x+3)^2 [4(x+3) + 3(2x-1)]$$

$$= (2x-1)(x+3)^2 (10x+9)$$

(3) $y = (x^2-x+1)^4(3x-1)^2$

[Sol] $y' = 4(x^2-x+1)^3(2x-1) \cdot (3x-1)^2 + (x^2-x+1)^4 \cdot 2(3x-1) \cdot 3$

$$= 2(x^2-x+1)^3(3x-1)[2(2x-1)(3x-1) + 3(x^2-x+1)]$$

$$= 2(x^2-x+1)^3(3x-1)(15x^2-13x+5)$$

#1

$$(4) \quad y = \left(\frac{x}{x+1} \right)^3$$

$$[\text{Sol}] \quad y' = 3 \left(\frac{x}{x+1} \right)^2 \cdot \frac{1 \cdot (x+1) - x \cdot 1}{(x+1)^2} \quad \leftarrow \quad \boxed{\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

$$= \frac{3x^2}{(x+1)^4}$$

$$(5) \quad y = \left(x + \frac{1}{x} \right)^3$$

$$[\text{Sol}] \quad y' = 3 \left(x + \frac{1}{x} \right)^2 \left(1 - \frac{1}{x^2} \right)$$

$$\left[= 3 \left(x + \frac{1}{x} \right)^2 \left(1 + \frac{1}{x} \right) \left(1 - \frac{1}{x} \right) \right] \quad \left[= \frac{3(x^2+1)^2(x+1)(x-1)}{x^4} \right]$$

$$(6) \quad y = \left(\frac{x^2+12}{x+2} \right)^4$$

$$[\text{Sol}] \quad y' = 4 \left(\frac{x^2+12}{x+2} \right)^3 \cdot \frac{2x(x+2) - (x^2+12) \cdot 1}{(x+2)^2}$$

$$= \frac{4(x^2+12)^3(x^2+4x-12)}{(x+2)^5} \quad \left[= \frac{4(x^2+12)^3(x+6)(x-2)}{(x+2)^5} \right]$$

$$(7) \quad y = \frac{(2x-1)^3}{(x^2+1)^2}$$

$$[\text{Sol}] \quad y' = \frac{3(2x-1)^2 \cdot 2 \cdot (x^2+1)^2 - (2x-1)^3 \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

$$= \frac{2(2x-1)^2 [3(x^2+1) - 2x(2x-1)]}{(x^2+1)^3}$$

$$= -\frac{2(2x-1)^2(x^2-2x-3)}{(x^2+1)^3} \quad \left[= -\frac{2(2x-1)^2(x-3)(x+1)}{(x^2+1)^3} \right]$$

Differentiation 2

Name _____

Date / /

Time : to :

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When the value of y for $y=f(x)$ is determined and then only one corresponding value of x is defined, x is considered to be a function of y . If this function is expressed as $x=g(y)$, then the function $y=g(x)$ where x and y are switched is called the **inverse function** of the original function $y=f(x)$.

Find the inverse functions of the following functions.

Ex. $y=2x+1$

[Sol] $x=\frac{1}{2}y-\frac{1}{2}$ ← Solving the equation for x

Switching x and y , $y=\frac{1}{2}x-\frac{1}{2}$

(1) $y=-3x+2$

[Sol] $x=-\frac{1}{3}y+\frac{2}{3}$

Switching x and y , $y=-\frac{1}{3}x+\frac{2}{3}$

(2) $y=3^x$

[Sol] $x=\log_3 y$

When $y=a^x$, $x=\log_a y$

Switching x and y , $y=\log_3 x$

(3) $y=\log_2 x$

[Sol] $x=2^y$

When $y=\log_a x$, $x=a^y$

Switching x and y , $y=2^x$

The inverse function of $f(x)$ is expressed as $f^{-1}(x)$. $f^{-1}(x)$ is read as "f inverse of x."
Therefore, if $y=f^{-1}(x)$, then $x=f(y)$.

Ex. $y = x^2 + 1 \quad (x \geq 1)$

[Sol] The range is $y \geq 2$. $\leftarrow$ Since $x \geq 1$

Since $x^2 = y - 1$, $x = \pm\sqrt{y-1}$

Since $x \geq 1$, $x = \sqrt{y-1}$

Switching x and y , $y = \sqrt{x-1} \quad (x \geq 2) \quad \leftarrow$

For the given function and its inverse function, the domain and the range are switched.

(4) $y = 1 - x^2 \quad (x \leq -1)$

[Sol] The range is $y \leq 0$.

Since $x^2 = 1 - y$, $x = \pm\sqrt{1-y}$

Since $x \leq -1$, $x = -\sqrt{1-y}$

Switching x and y , $y = -\sqrt{1-x} \quad (x \leq 0)$

(5) $y = 3x - 2 \quad (0 \leq x \leq 4)$

[Sol] The range is $-2 \leq y \leq 10$.

$$x = \frac{1}{3}y + \frac{2}{3}$$

Switching x and y , $y = \frac{1}{3}x + \frac{2}{3} \quad (-2 \leq x \leq 10)$

(6) $y = \sqrt{x+1}$

[Sol] The range is $y \geq 0$.

Since $y^2 = x + 1$, $x = y^2 - 1$

Switching x and y , $y = x^2 - 1 \quad (x \geq 0)$

Differentiation 2

Name _____

Date / /

Time : to :

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Prove $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ if $\frac{dx}{dy} \neq 0$.

[Sol] Differentiating both sides of $x = f(y)$ with respect to x ,

$$\text{LHS} = \frac{d}{dx} x = 1$$

$$\text{RHS} = \frac{d}{dx} f(y) = \frac{d}{dy} f(y) \cdot \frac{dy}{dx} = \boxed{\frac{dx}{dy}} \cdot \frac{dy}{dx}$$

$$\therefore \boxed{\frac{dx}{dy}} \cdot \frac{dy}{dx} = 1$$

Therefore, if $\frac{dx}{dy} \neq 0$, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Answers: All the answers are the same, $\frac{dy}{dx}$

Differentiation Formula for Inverse Functions

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \left(\frac{dx}{dy} \neq 0 \right)$$

Using the formula above, express $\frac{dy}{dx}$ in terms of x .

Ex. $y = \sqrt[3]{x}$

[Sol] Since $x = y^3$, $\frac{dx}{dy} = 3y^2$ ←

Rearranging into the form $x = \bullet$, then differentiating with respect to y

$$\therefore \frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3\sqrt[3]{x^2}} \leftarrow$$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, then rewriting in terms of x

(1) $y = \sqrt[4]{x}$

[Sol] Since $x = y^4$, $\frac{dx}{dy} = 4y^3$

$$\therefore \frac{dy}{dx} = \frac{1}{4y^3} = \frac{1}{4\sqrt[4]{x^3}}$$

N155b

(2) $y = \sqrt{x}$

[Sol] Since $x = y^2$, $\frac{dx}{dy} = 2y$

$$\therefore \frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

(3) $y = x^{\frac{1}{6}}$

[Sol] Since $x = y^6$, $\frac{dx}{dy} = 6y^5$

$$\therefore \frac{dy}{dx} = \frac{1}{6y^5} = \frac{1}{6x^{\frac{5}{6}}} \quad \left[= \frac{1}{6} x^{-\frac{5}{6}} \right]$$

(4) $x = y^5$

[Sol] $\frac{dx}{dy} = 5y^4$

Since $y^5 = x$, $y = x^{\frac{1}{5}}$ $\leftarrow$ Solving the equation for y

$$\therefore \frac{dy}{dx} = \frac{1}{5y^4} = \frac{1}{5x^{\frac{4}{5}}} \quad \left[= \frac{1}{5} x^{-\frac{4}{5}} \right] \quad \left[= \frac{1}{5\sqrt[5]{x^4}} \right]$$

(5) $x = y^2 + y + 1 \quad \left(y > -\frac{1}{2} \right)$

[Sol] $\frac{dx}{dy} = 2y + 1$

Since $y^2 + y + 1 - x = 0$, $y = \frac{-1 \pm \sqrt{4x-3}}{2}$ $\leftarrow$

Since $y > -\frac{1}{2}$, $y = \frac{-1 + \sqrt{4x-3}}{2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2y+1} = \frac{1}{2 \cdot \frac{-1 + \sqrt{4x-3}}{2} + 1} = \frac{1}{\sqrt{4x-3}}$$

Considering x as a constant and solving the quadratic equation for y (Quadratic Formula J97)

Differentiation 2

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In N148, it has been proved that $(x^n)' = nx^{n-1}$ is true when n is an integer. Here, consider if it is also true when n is a rational number.

Differentiate function $y = x^{\frac{m}{n}}$. (m and n ($n > 0$) are integers.)

[Sol] Raising both sides to the n^{th} power, $y^n = x^m$

Differentiating both sides with respect to x ,

$$\text{LHS} = \frac{d}{dx} y^n = \frac{d}{dy} y^n \cdot \frac{dy}{dx} = ny^{n-1} \cdot \frac{dy}{dx}$$

$$\text{RHS} = mx^{m-1}$$

$$\therefore ny^{n-1} \cdot \frac{dy}{dx} = mx^{m-1}$$

Therefore,

$$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}} = \frac{mx^{m-1}}{n(x^{\frac{m}{n}})^{n-1}} = \frac{mx^{m-1}}{nx^{\frac{m}{n} \cdot (n-1)}} = \frac{m}{n} x^{\frac{m}{n} - 1}$$

From the above, $(x^n)' = nx^{n-1}$ is also true when n is a rational number.

Derivative of x^p

When p is a rational number, $(x^p)' = px^{p-1}$

N156b

Differentiate the following functions.

Ex.

$$y = x^{\frac{1}{2}}$$

$$[\text{Sol}] \quad y' = \frac{1}{2}x^{-\frac{1}{2}}$$

$$(1) \quad y = x^{\frac{2}{3}}$$

$$[\text{Sol}] \quad y' = \frac{2}{3}x^{-\frac{1}{3}}$$

$$(2) \quad y = x^{\frac{5}{2}}$$

$$[\text{Sol}] \quad y' = \frac{5}{2}x^{\frac{3}{2}}$$

$$(3) \quad y = x^{-\frac{1}{2}}$$

$$[\text{Sol}] \quad y' = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$(4) \quad y = x^{-\frac{3}{2}}$$

$$[\text{Sol}] \quad y' = -\frac{3}{2}x^{-\frac{5}{2}}$$

Ex.

$$y = \sqrt[3]{x^2}$$

$$[\text{Sol}] \quad y = x^{\frac{2}{3}}$$



Rearranging into the form $y = x^a$

$$y' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$(5) \quad y = \sqrt[5]{x^3}$$

$$[\text{Sol}] \quad y = x^{\frac{3}{5}}$$

$$y' = \frac{3}{5}x^{-\frac{2}{5}} = \frac{3}{5\sqrt[5]{x^2}}$$

$$(7) \quad y = \frac{1}{x^4\sqrt{x}}$$

$$[\text{Sol}] \quad y = x^{-\frac{5}{4}}$$

$$y' = -\frac{5}{4}x^{-\frac{9}{4}} = -\frac{5}{4x^2\sqrt[4]{x}}$$

$$(6) \quad y = \frac{1}{\sqrt[3]{x^2}}$$

$$[\text{Sol}] \quad y = x^{-\frac{2}{3}}$$

$$y' = -\frac{2}{3}x^{-\frac{5}{3}} = -\frac{2}{3x\sqrt[3]{x^2}}$$

$$(8) \quad y = \sqrt{x}\sqrt{x}$$

$$[\text{Sol}] \quad y = \left[x \cdot \left(x^{\frac{1}{2}} \right) \right]^{\frac{1}{2}} = \left(x^{\frac{3}{2}} \right)^{\frac{1}{2}} = x^{\frac{3}{4}}$$

$$y' = \frac{3}{4}x^{-\frac{1}{4}} = \frac{3}{4\sqrt[4]{x}}$$

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Differentiate the following functions.

Ex. $y = \sqrt{3x+4}$

[Sol] $y = (3x+4)^{\frac{1}{2}}$

$$y' = \frac{1}{2}(3x+4)^{-\frac{1}{2}} \cdot 3 = \frac{3}{2\sqrt{3x+4}}$$

← $y = \square^{\frac{1}{2}}$

← $y' = \frac{1}{2} \square^{-\frac{1}{2}} \cdot \square'$

(1) $y = \sqrt[3]{1-4x}$

[Sol] $y = (1-4x)^{\frac{1}{3}}$

$$y' = \frac{1}{3}(1-4x)^{-\frac{2}{3}} \cdot (-4) = -\frac{4}{3\sqrt[3]{(1-4x)^2}}$$

(2) $y = \sqrt{2x^2-3x+5}$

[Sol] $y = (2x^2-3x+5)^{\frac{1}{2}}$

$$y' = \frac{1}{2}(2x^2-3x+5)^{-\frac{1}{2}} \cdot (4x-3) = \frac{4x-3}{2\sqrt{2x^2-3x+5}}$$

(3) $y = \frac{5}{\sqrt[3]{3x-4}}$

[Sol] $y = 5(3x-4)^{-\frac{1}{3}}$

$$y' = -\frac{5}{3}(3x-4)^{-\frac{4}{3}} \cdot 3 = -\frac{5}{(3x-4)\sqrt[3]{3x-4}}$$

N157b

$$(4) \quad y = (\sqrt{x} + 1)^2$$

$$[\text{Sol}] \quad y = \left(x^{\frac{1}{2}} + 1\right)^2$$

$$y' = 3\left(x^{\frac{1}{2}} + 1\right)^2 \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{3(\sqrt{x} + 1)^2}{2\sqrt{x}}$$

$$(5) \quad y = \sqrt{(4 - x^2)^3}$$

$$[\text{Sol}] \quad y = (4 - x^2)^{\frac{3}{2}}$$

$$y' = \frac{3}{2}(4 - x^2)^{\frac{1}{2}} \cdot (-2x) = -3x\sqrt{4 - x^2}$$

$$(6) \quad y = (x + \sqrt{x^2 + 1})^4$$

$$[\text{Sol}] \quad y = \left[x + (x^2 + 1)^{\frac{1}{2}}\right]^4$$

$$y' = 4\left[x + (x^2 + 1)^{\frac{1}{2}}\right]^3 \cdot \left[1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x\right] \quad \leftarrow$$

$$= 4(x + \sqrt{x^2 + 1})^3 \cdot \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}$$

$$= \frac{4(x + \sqrt{x^2 + 1})^4}{\sqrt{x^2 + 1}}$$

$$\left[x + (x^2 + 1)^{\frac{1}{2}}\right] \\ = 1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot (x^2 + 1)'$$

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Differentiate the following functions.

(1) $y = (3x-1)\sqrt{x+2}$

[Sol] $y = (3x-1)(x+2)^{\frac{1}{2}}$

$$y' = 3(x+2)^{\frac{1}{2}} + (3x-1) \cdot \frac{1}{2}(x+2)^{-\frac{1}{2}} \quad \leftarrow$$

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$= 3\sqrt{x+2} + \frac{3x-1}{2\sqrt{x+2}}$$

$$= \frac{9x+11}{2\sqrt{x+2}}$$

(2) $y = (x+4)\sqrt{x^2+4}$

[Sol] $y = (x+4)(x^2+4)^{\frac{1}{2}}$

$$y' = 1 \cdot (x^2+4)^{\frac{1}{2}} + (x+4) \cdot \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \cdot 2x$$

$$= \sqrt{x^2+4} + \frac{x(x+4)}{\sqrt{x^2+4}}$$

$$= \frac{2x^2+4x+4}{\sqrt{x^2+4}} \quad \left[= \frac{2(x^2+2x+2)}{\sqrt{x^2+4}} \right]$$

(3) $y = (2x+1)\sqrt{2-x^2}$

[Sol] $y = (2x+1)(2-x^2)^{\frac{1}{2}}$

$$y' = 2(2-x^2)^{\frac{1}{2}} + (2x+1) \cdot \frac{1}{2}(2-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= 2\sqrt{2-x^2} - \frac{x(2x+1)}{\sqrt{2-x^2}}$$

$$= -\frac{4x^2+x-4}{\sqrt{2-x^2}}$$

$$(4) \quad y = \frac{x}{\sqrt{x+1}}$$

$$[\text{Sol}] \quad y = x(x+1)^{-\frac{1}{2}}$$

$$\begin{aligned} y' &= 1 \cdot (x+1)^{-\frac{1}{2}} + x \cdot \left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}} \\ &= \frac{1}{\sqrt{x+1}} - \frac{x}{2(x+1)\sqrt{x+1}} \\ &= \frac{x+2}{2(x+1)\sqrt{x+1}} \quad \left[= \frac{(x+2)\sqrt{x+1}}{2(x+1)^2} \right] \end{aligned}$$

Alternative Solution

$$\begin{aligned} y' &= \frac{1 \cdot \sqrt{x+1} - x \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}}}{x+1} \\ &= \frac{2(x+1) - x}{2(x+1)\sqrt{x+1}} \\ &= \frac{x+2}{2(x+1)\sqrt{x+1}} \\ &= \frac{(x+2)\sqrt{x+1}}{2(x+1)^2} \end{aligned}$$

$$(5) \quad y = \frac{x}{\sqrt{2x^2+1}}$$

$$[\text{Sol}] \quad y = x(2x^2+1)^{-\frac{1}{2}}$$

$$\begin{aligned} y' &= 1 \cdot (2x^2+1)^{-\frac{1}{2}} + x \cdot \left(-\frac{1}{2}\right)(2x^2+1)^{-\frac{3}{2}} \cdot 4x \\ &= \frac{1}{\sqrt{2x^2+1}} - \frac{2x^2}{(2x^2+1)\sqrt{2x^2+1}} \\ &= \frac{1}{(2x^2+1)\sqrt{2x^2+1}} \quad \left[= \frac{\sqrt{2x^2+1}}{(2x^2+1)^2} \right] \end{aligned}$$

Alternative Solution

$$\begin{aligned} y' &= \frac{1 \cdot \sqrt{2x^2+1} - x \cdot \frac{1}{2}(2x^2+1)^{-\frac{1}{2}} \cdot 4x}{2x^2+1} \\ &= \frac{(2x^2+1) - 2x^2}{(2x^2+1)\sqrt{2x^2+1}} \\ &= \frac{1}{(2x^2+1)\sqrt{2x^2+1}} \\ &= \frac{\sqrt{2x^2+1}}{(2x^2+1)^2} \end{aligned}$$

$$(6) \quad y = \sqrt{\frac{x^2+1}{x+1}}$$

$$[\text{Sol}] \quad y = (x^2+1)^{\frac{1}{2}}(x+1)^{-\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x \cdot (x+1)^{-\frac{1}{2}} + (x^2+1)^{\frac{1}{2}} \cdot \left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}} \\ &= \frac{x}{\sqrt{x^2+1}\sqrt{x+1}} - \frac{\sqrt{x^2+1}}{2(x+1)\sqrt{x+1}} \\ &= \frac{x^2+2x-1}{2(x+1)\sqrt{x+1}\sqrt{x^2+1}} \quad \left[= \frac{(x^2+2x-1)\sqrt{x+1}}{2(x+1)^2\sqrt{x^2+1}} \right] \end{aligned}$$

Alternative Solution

$$\begin{aligned} y &= \left(\frac{x^2+1}{x+1}\right)^{\frac{1}{2}}, \quad y' = \frac{1}{2} \left(\frac{x^2+1}{x+1}\right)^{-\frac{1}{2}} \cdot \frac{2x(x+1) - (x^2+1) \cdot 1}{(x+1)^2} \\ &= \frac{1}{2} \sqrt{\frac{x+1}{x^2+1}} \cdot \frac{x^2+2x-1}{(x+1)^2} \\ &= \frac{(x^2+2x-1)\sqrt{x+1}}{2(x+1)^2\sqrt{x^2+1}} \quad \left[= \frac{x^2+2x-1}{2(x+1)\sqrt{x+1}\sqrt{x^2+1}} \right] \end{aligned}$$

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1. Given that $f(x) = \sqrt{ax^2 - 1}$ satisfies $f'(1) = 2$, find the value of constant a .

[Sol] $f(x) = (ax^2 - 1)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(ax^2 - 1)^{-\frac{1}{2}} \cdot 2ax$$

$$= \frac{ax}{\sqrt{ax^2 - 1}}$$

Since $f'(1) = 2$, $\frac{a}{\sqrt{a-1}} = 2$

$$a^2 - 4a + 4 = 0$$

$$(a-2)^2 = 0$$

$$\therefore a = 2$$

Squaring both sides and simplifying

2. Show that $(y-x)y' = y$ when $y = x + \sqrt{x^2 + a^2}$.

[Sol] $y = x + (x^2 + a^2)^{\frac{1}{2}}$

$$y' = 1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}$$

$$\therefore (y-x)y' = \sqrt{x^2 + a^2} \cdot \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}$$

$$= x + \sqrt{x^2 + a^2}$$

$$= y$$

$$\therefore (y-x)y' = y$$

N159b

3. Given $f(x) = \lim_{t \rightarrow x} \frac{t\sqrt{x-1} - x\sqrt{t-1}}{t-x}$, solve the following questions. ($1 < x$)

(1) Find $f(x)$.

$$[\text{Sol}] f(x) = \lim_{t \rightarrow x} \frac{(t\sqrt{x-1} - x\sqrt{t-1})(t\sqrt{x-1} + x\sqrt{t-1})}{(t-x)(t\sqrt{x-1} + x\sqrt{t-1})}$$

$$= \lim_{t \rightarrow x} \frac{(t-x)(tx-t-x)}{(t-x)(t\sqrt{x-1} + x\sqrt{t-1})}$$

$$= \lim_{t \rightarrow x} \frac{tx-t-x}{t\sqrt{x-1} + x\sqrt{t-1}}$$

$$= \frac{x^2-2x}{2x\sqrt{x-1}}$$

$$= \frac{x-2}{2\sqrt{x-1}}$$

$$\begin{aligned} & (t\sqrt{x-1})^2 - (x\sqrt{t-1})^2 \\ &= t^2(x-1) - x^2(t-1) \\ &= tx(t-x) - (t+x)(t-x) \\ &= (t-x)(tx-t-x) \end{aligned}$$

(2) Find $f'(x)$.

$$[\text{Sol}] \text{ From (1), } f(x) = \frac{1}{2}(x-2)(x-1)^{-\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} \left[1 \cdot (x-1)^{-\frac{1}{2}} + (x-2) \cdot \left(-\frac{1}{2} \right) (x-1)^{-\frac{3}{2}} \right]$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{x-1}} - \frac{x-2}{2(x-1)\sqrt{x-1}} \right]$$

$$= \frac{x}{4(x-1)\sqrt{x-1}} \quad \left[= \frac{x\sqrt{x-1}}{4(x-1)^2} \right]$$

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Differentiate the following functions.

(1) $y = (3x^2 - 5)^4$ ➡ N

[Sol] $y' = 4(3x^2 - 5)^3 \cdot 6x$
 $= 24x(3x^2 - 5)^3$

(2) $y = \frac{1}{(2-x)^3}$ ➡ N

[Sol] $y = (2-x)^{-3}$
 $y' = -3(2-x)^{-4} \cdot (-1)$
 $= \frac{3}{(2-x)^4}$

Alternative Solution

$$y' = -\frac{3(2-x)^2 \cdot (-1)}{(2-x)^6}$$

$$= \frac{3}{(2-x)^4}$$

(3) $y = \left(\frac{x}{3x+1}\right)^4$ ➡ N

[Sol] $y' = 4\left(\frac{x}{3x+1}\right)^3 \cdot \frac{1 \cdot (3x+1) - x \cdot 3}{(3x+1)^2}$
 $= \frac{4x^3}{(3x+1)^5}$

N160b

$$(4) \quad y = \sqrt[3]{4x+1}$$

⇒ N157

$$[\text{Sol}] \quad y = (4x+1)^{\frac{1}{3}}$$

$$\begin{aligned} y' &= \frac{1}{3}(4x+1)^{-\frac{2}{3}} \cdot 4 \\ &= \frac{4}{3\sqrt[3]{(4x+1)^2}} \end{aligned}$$

$$(5) \quad y = x\sqrt{1+x^2}$$

⇒ N158

$$[\text{Sol}] \quad y = x(1+x^2)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= 1 \cdot (1+x^2)^{\frac{1}{2}} + x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x \\ &= \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} \\ &= \frac{1+2x^2}{\sqrt{1+x^2}} \end{aligned}$$

$$(6) \quad y = \frac{x}{\sqrt{x^2+1}}$$

⇒ N158

$$[\text{Sol}] \quad y = x(x^2+1)^{-\frac{1}{2}}$$

$$\begin{aligned} y' &= 1 \cdot (x^2+1)^{-\frac{1}{2}} + x \cdot \left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}} \cdot 2x \\ &= \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)\sqrt{x^2+1}} \\ &= \frac{1}{(x^2+1)\sqrt{x^2+1}} \left[= \frac{\sqrt{x^2+1}}{(x^2+1)^2} \right] \end{aligned}$$

Alternative Solution

$$\left[\frac{1 \cdot \sqrt{x^2+1} - x \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x}{x^2+1} = \frac{(x^2+1) - x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+1)\sqrt{x^2+1}} \left[= \frac{\sqrt{x^2+1}}{(x^2+1)^2} \right] \right]$$

Differentiation of Trigonometric Functions

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[1] Using the definition of the derivative, differentiate $y = \sin x$.

[Sol] $y' = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h \rightarrow 0} \frac{2 \cos \frac{(x+h)+x}{2} \sin \frac{(x+h)-x}{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2 \cos \left(x + \frac{h}{2} \right) \sin \frac{h}{2}}{h}$$

$$= \lim_{h \rightarrow 0} \cos \left(x + \frac{h}{2} \right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}}$$

$$= \cos x \cdot 1$$

$$= \cos x$$

$$\begin{aligned} \sin A - \sin B &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\sin x = \cos \left(\frac{\pi}{2} - x \right) \quad \cos x = \sin \left(\frac{\pi}{2} - x \right)$$

The Sum-to-Product Formulas introduced in M177 are often used for the differentiation and integration of trigonometric functions.

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

[II] Using the definition of the derivative, differentiate $y = \cos x$.

[Sol] $y' = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$

$$= \lim_{h \rightarrow 0} \frac{-2 \sin \frac{(x+h)+x}{2} \sin \frac{(x+h)-x}{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2 \sin \left(x + \frac{h}{2} \right) \sin \frac{h}{2}}{h}$$

$$= \lim_{h \rightarrow 0} \left[-\sin \left(x + \frac{h}{2} \right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right]$$

$$= -\sin x \cdot 1$$

$$= -\sin x$$

$$\begin{aligned} \cos A - \cos B \\ = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Answers: $\frac{h}{2}, \frac{2}{h}, -\sin x, -\sin x$

[III] Differentiate $y = \tan x$.

[Sol] $y' = \left(\frac{\sin x}{\cos x} \right)'$

$$= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

From [I] and [II]

Answers: $\sin x, \sin x, \cos^2 x + \sin^2 x, 1$

Differentiation of Trigonometric Functions

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From the results in N161, the following formulas are true.

Derivatives of Trigonometric Functions

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \frac{1}{\cos^2 x}$$

Differentiate the following functions.

Ex. $y = x \sin x$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x)' \sin x + x (\sin x)' \\ &= 1 \cdot \sin x + x \cos x \\ &= \sin x + x \cos x \end{aligned}$$

$$\begin{aligned} [f(x)g(x)]' &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

(1) $y = x \cos x$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x)' \cos x + x (\cos x)' \\ &= 1 \cdot \cos x - x \sin x \\ &= \cos x - x \sin x \end{aligned}$$

(2) $y = x \tan x$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x)' \tan x + x (\tan x)' \\ &= 1 \cdot \tan x + x \cdot \frac{1}{\cos^2 x} \\ &= \tan x + \frac{x}{\cos^2 x} \quad \left[= \frac{\sin x \cos x + x}{\cos^2 x} \right] \end{aligned}$$

(3) $y = (3x-2) \sin x$

$$\begin{aligned} \text{[Sol]} \quad y' &= (3x-2)' \sin x + (3x-2) (\sin x)' \\ &= 3 \sin x + (3x-2) \cos x \end{aligned}$$

N162b

$$(4) \quad y = (2x^2 + 1) \cos x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (2x^2 + 1)' \cos x + (2x^2 + 1)(\cos x)' \\ &= 4x \cos x - (2x^2 + 1) \sin x \end{aligned}$$

$$(5) \quad y = \sin x \cos x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (\sin x)' \cos x + \sin x (\cos x)' \\ &= \cos^2 x - \sin^2 x \\ &= 2 \cos^2 x - 1 \\ &= 1 - 2 \sin^2 x \\ &= \cos 2x \end{aligned}$$

$$(6) \quad y = \sin x \tan x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (\sin x)' \tan x + \sin x (\tan x)' \\ &= \cos x \cdot \frac{\sin x}{\cos x} + \sin x \cdot \frac{1}{\cos^2 x} \\ &= \sin x + \frac{\sin x}{\cos^2 x} \quad \left[= \frac{\sin x (\cos^2 x + 1)}{\cos^2 x} \right] \end{aligned}$$

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(mistakes) 0	—	1	2	3~

Differentiate the following functions.

Ex.

$$y = \cos 2x$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= -\sin 2x \cdot (2x)' \quad \leftarrow [f(g(x))]' = f'(g(x))g'(x) \\
 &= -2\sin 2x
 \end{aligned}$$

(1) $y = \sin 5x$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \cos 5x \cdot (5x)' \\
 &= 5\cos 5x
 \end{aligned}$$

(2) $y = \tan 3x + \cos 4x$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \frac{1}{\cos^2 3x} \cdot (3x)' - \sin 4x \cdot (4x)' \\
 &= \frac{3}{\cos^2 3x} - 4\sin 4x
 \end{aligned}$$

(3) $y = \sin(2x-3)$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \cos(2x-3) \cdot (2x-3)' \\
 &= 2\cos(2x-3)
 \end{aligned}$$

(4) $y = \cos(3-2x^2)$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= -\sin(3-2x^2) \cdot (3-2x^2)' \\
 &= 4x\sin(3-2x^2)
 \end{aligned}$$

N163b

(5) $y = \tan(x^3 + 1)$

[Sol] $y' = \frac{1}{\cos^2(x^3 + 1)} \cdot (x^3 + 1)'$
 $= \frac{3x^2}{\cos^2(x^3 + 1)}$

(6) $y = \cos\left(\frac{x}{2} + \frac{\pi}{6}\right)$

[Sol] $y' = -\sin\left(\frac{x}{2} + \frac{\pi}{6}\right) \cdot \left(\frac{x}{2} + \frac{\pi}{6}\right)'$
 $= -\frac{1}{2} \sin\left(\frac{x}{2} + \frac{\pi}{6}\right)$

(7) $y = \tan\left(\frac{\pi}{3} - \pi x\right)$

[Sol] $y' = \frac{1}{\cos^2\left(\frac{\pi}{3} - \pi x\right)} \cdot \left(\frac{\pi}{3} - \pi x\right)'$
 $= -\frac{\pi}{\cos^2\left(\frac{\pi}{3} - \pi x\right)}$

(8) $y = x \sin(3x + 5)$

[Sol] $y' = (x)' \sin(3x + 5) + x [\sin(3x + 5)]'$
 $= 1 \cdot \sin(3x + 5) + x \cos(3x + 5) \cdot (3x + 5)'$
 $= \sin(3x + 5) + 3x \cos(3x + 5)$

Differentiation of Trigonometric Functions

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	—	2~

Differentiate the following functions.

(1) $y = \sin \frac{1}{x}$

[Sol] $y' = \cos \frac{1}{x} \cdot \left(\frac{1}{x} \right)'$
 $= -\frac{1}{x^2} \cos \frac{1}{x}$

(2) $y = \cos \frac{1}{x^2}$

[Sol] $y' = -\sin \frac{1}{x^2} \cdot \left(\frac{1}{x^2} \right)'$
 $= \frac{2}{x^3} \sin \frac{1}{x^2}$

$\left(\frac{1}{x^2} \right)' = (x^{-2})' = -2x^{-3} = -\frac{2}{x^3}$

(3) $y = \sin \frac{x-1}{x+1}$

[Sol] $y' = \cos \frac{x-1}{x+1} \cdot \left(\frac{x-1}{x+1} \right)'$
 $= \cos \frac{x-1}{x+1} \cdot \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2}$
 $= \frac{2}{(x+1)^2} \cos \frac{x-1}{x+1}$

N164b

(4) $y = \cos \sqrt{x}$

[Sol] $y' = -\sin \sqrt{x} \cdot (\sqrt{x})'$

$$= -\sin \sqrt{x} \cdot \frac{1}{2} x^{\frac{1}{2}}$$

$$(\sqrt{x})' = (x^{\frac{1}{2}})'$$

$$= -\frac{\sin \sqrt{x}}{2\sqrt{x}}$$

(5) $y = \sin \sqrt{x^2 + x + 1}$

[Sol] $y' = \cos \sqrt{x^2 + x + 1} \cdot (\sqrt{x^2 + x + 1})'$

$$= \cos \sqrt{x^2 + x + 1} \cdot \frac{1}{2} (x^2 + x + 1)^{-\frac{1}{2}} \cdot (2x + 1)$$

$$= \frac{(2x + 1) \cos \sqrt{x^2 + x + 1}}{2\sqrt{x^2 + x + 1}}$$

$$\begin{aligned} (\sqrt{x^2 + x + 1})' &= [(x^2 + x + 1)^{\frac{1}{2}}]' \\ &= \frac{1}{2} (x^2 + x + 1)^{-\frac{1}{2}} \cdot (x^2 + x + 1)' \end{aligned}$$

(6) $y = \tan(\sin x)$

[Sol] $y' = \frac{1}{\cos^2(\sin x)} \cdot (\sin x)'$

$$= \frac{\cos x}{\cos^2(\sin x)}$$

Differentiation of
Trigonometric Functions

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	—	2~

Differentiate the following functions.

Ex.

$y = \sin^3 x$



$y = \blacksquare^3$

[Sol] $y' = 3\sin^2 x \cdot (\sin x)'$



$y' = 3\blacksquare^2 \cdot \blacksquare'$

$= 3\sin^2 x \cos x$

(1) $y = \cos^3 x$

[Sol] $y' = 3\cos^2 x \cdot (\cos x)'$

$= -3\cos^2 x \sin x$

(2) $y = \tan^2 x$

[Sol] $y' = 2\tan x \cdot (\tan x)'$

$= 2\tan x \cdot \frac{1}{\cos^2 x}$

$= \frac{2\tan x}{\cos^2 x} \quad \left[= \frac{2\sin x}{\cos^3 x} \right]$

(3) $y = (\sin x + \cos x)^3$

[Sol] $y' = 3(\sin x + \cos x)^2 \cdot (\sin x + \cos x)'$

$= 3(\sin x + \cos x)^2 (\cos x - \sin x)$

N165b

$$(4) \quad y = \sin^4(3x+2)$$

$$[\text{Sol}] \quad y' = 4\sin^3(3x+2) \cdot [\sin(3x+2)]'$$

$$= 4\sin^3(3x+2) \cos(3x+2) \cdot (3x+2)'$$

$$= 12\sin^3(3x+2) \cos(3x+2)$$

$$[f(g(x))]' = f'(g(x))g'(x)$$

$$(5) \quad y = \cos^3(1-2x^2)$$

$$[\text{Sol}] \quad y' = 3\cos^2(1-2x^2) \cdot [\cos(1-2x^2)]'$$

$$= 3\cos^2(1-2x^2) \cdot [-\sin(1-2x^2)] \cdot (1-2x^2)'$$

$$= 12x \cos^2(1-2x^2) \sin(1-2x^2)$$

$$(6) \quad y = x^2 \sin^3(2x-5)$$

$$[\text{Sol}] \quad y' = (x^2)' \sin^3(2x-5) + x^2 [\sin^3(2x-5)]'$$

$$= 2x \sin^3(2x-5) + x^2 \cdot 3\sin^2(2x-5) \cdot [\sin(2x-5)]'$$

$$= 2x \sin^3(2x-5) + 3x^2 \sin^2(2x-5) \cos(2x-5) \cdot (2x-5)'$$

$$= 2x \sin^3(2x-5) + 6x^2 \sin^2(2x-5) \cos(2x-5)$$

$$[= 2x \sin^2(2x-5) [\sin(2x-5) + 3x \cos(2x-5)]]$$

Differentiation of Trigonometric Functions

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	—	2~

Differentiate the following functions.

Ex.

$$y = \sqrt{\cos 3x}$$

$$[\text{Sol}] \quad y = (\cos 3x)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2} (\cos 3x)^{-\frac{1}{2}} \cdot (\cos 3x)' \\ &= -\frac{3 \sin 3x}{2\sqrt{\cos 3x}} \end{aligned}$$

Rearranging into the form $y = \blacksquare^{\blacksquare}$

$$y' = \frac{1}{2} \blacksquare^{-\frac{1}{2}} \cdot \blacksquare'$$

$$(\cos 3x)' = -\sin 3x \cdot (3x)'$$

$$(1) \quad y = \sqrt{\sin x}$$

$$[\text{Sol}] \quad y = (\sin x)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot (\sin x)' \\ &= \frac{\cos x}{2\sqrt{\sin x}} \end{aligned}$$

$$(2) \quad y = \sqrt{1 - \sin x}$$

$$[\text{Sol}] \quad y = (1 - \sin x)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2} (1 - \sin x)^{-\frac{1}{2}} \cdot (1 - \sin x)' \\ &= -\frac{\cos x}{2\sqrt{1 - \sin x}} \end{aligned}$$

N166b

$$(3) \quad y = \sqrt{\cos 2x - x}$$

$$[\text{Sol}] \quad y = (\cos 2x - x)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2}(\cos 2x - x)^{-\frac{1}{2}} \cdot (\cos 2x - x)' \\ &= -\frac{2\sin 2x + 1}{2\sqrt{\cos 2x - x}} \end{aligned}$$

$$(4) \quad y = \sqrt{1 + \sin^2 x}$$

$$[\text{Sol}] \quad y = (1 + \sin^2 x)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2}(1 + \sin^2 x)^{-\frac{1}{2}} \cdot (1 + \sin^2 x)' \\ &= \frac{2\sin x \cdot (\sin x)'}{2\sqrt{1 + \sin^2 x}} \\ &= \frac{\sin x \cos x}{\sqrt{1 + \sin^2 x}} \end{aligned}$$

$$(5) \quad y = \frac{2}{\sqrt{\tan x}}$$

$$[\text{Sol}] \quad y = 2(\tan x)^{-\frac{1}{2}}$$

$$\begin{aligned} y' &= 2 \cdot \left(-\frac{1}{2}\right) (\tan x)^{-\frac{3}{2}} \cdot (\tan x)' \\ &= -\frac{1}{\tan x \sqrt{\tan x}} \cdot \frac{1}{\cos^2 x} \\ &= -\frac{1}{\sin x \cos x \sqrt{\tan x}} \\ &= -\frac{1}{\sqrt{\sin^3 x \cos x}} \end{aligned}$$

Differentiation of Trigonometric Functions

Name _____

Date / /

Time : to :

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(mistake) 0	—	1	—	2~

Differentiate the following functions.

Ex.

$$y = \frac{1}{\tan x}$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= -\frac{(\tan x)'}{\tan^2 x} \quad \leftarrow \quad \left[\frac{1}{g(x)} \right]' = -\frac{g'(x)}{[g(x)]^2} \\
 &= -\frac{1}{\tan^2 x \cos^2 x} \\
 &= -\frac{1}{\sin^2 x}
 \end{aligned}$$

$$(1) \quad y = \frac{1}{\cos x}$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= -\frac{(\cos x)'}{\cos^2 x} \\
 &= \frac{\sin x}{\cos^2 x}
 \end{aligned}$$

$$(2) \quad y = \frac{1}{\sin^2 x}$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= -\frac{(\sin^2 x)'}{\sin^4 x} \\
 &= -\frac{2 \sin x \cdot (\sin x)'}{\sin^4 x} \\
 &= -\frac{2 \cos x}{\sin^3 x}
 \end{aligned}$$

N167b

$$(3) \quad y = \frac{1}{\tan 3x}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= -\frac{(\tan 3x)'}{\tan^2 3x} \\ &= -\frac{3}{\tan^2 3x \cos^2 3x} \\ &= -\frac{3}{\sin^2 3x} \end{aligned}$$

$$(4) \quad y = \frac{\cos x}{x}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(\cos x)' \cdot x - \cos x \cdot (x)'}{x^2} \quad \leftarrow \left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{-x \sin x - \cos x \cdot 1}{x^2} \\ &= -\frac{x \sin x + \cos x}{x^2} \end{aligned}$$

$$(5) \quad y = \frac{\tan x}{\cos x}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(\tan x)' \cos x - \tan x (\cos x)'}{\cos^2 x} \\ &= \frac{\frac{1}{\cos^2 x} \cdot \cos x + \frac{\sin x}{\cos x} \cdot \sin x}{\cos^2 x} \\ &= \frac{1 + \sin^2 x}{\cos^3 x} \end{aligned}$$

Differentiation of
Trigonometric Functions

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	—	2~

Differentiate the following functions.

$$(1) \quad y = \frac{\cos x}{1 + \cos x}$$

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(\cos x)'(1 + \cos x) - \cos x(1 + \cos x)'}{(1 + \cos x)^2} \\
 &= \frac{-\sin x(1 + \cos x) + \cos x \sin x}{(1 + \cos x)^2} \\
 &= -\frac{\sin x}{(1 + \cos x)^2}
 \end{aligned}$$

$$(2) \quad y = \frac{\sin x - \cos x}{\sin x + \cos x}$$

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(\sin x - \cos x)'(\sin x + \cos x) - (\sin x - \cos x)(\sin x + \cos x)'}{(\sin x + \cos x)^2} \\
 &= \frac{(\sin x + \cos x)^2 + (\sin x - \cos x)^2}{(\sin x + \cos x)^2} \\
 &= \frac{2}{(\sin x + \cos x)^2}
 \end{aligned}$$

$$(3) \quad y = \sin x \cos^2 x$$

$$\begin{aligned}
 [\text{Sol}] \quad y' &= (\sin x)' \cos^2 x + \sin x (\cos^2 x)' \\
 &= \cos^2 x + \sin x \cdot 2 \cos x \cdot (\cos x)' \\
 &= \cos^2 x - 2 \sin^2 x \cos x \\
 &= \cos^2 x - 2(1 - \cos^2 x) \cos x \\
 &= 3 \cos^2 x - 2 \cos x \\
 &= \cos x (3 \cos^2 x - 2) \\
 &= \cos x (1 - 3 \sin^2 x)
 \end{aligned}$$

Alternative Solution

$$\begin{aligned}
 y &= \sin x (1 - \sin^2 x) \\
 &= \sin x - \sin^3 x \\
 y' &= \cos x - 3 \sin^2 x \cdot (\sin x)' \\
 &= \cos x - 3 \sin^2 x \cos x \\
 &= \cos x - 3(1 - \cos^2 x) \cos x \\
 &= 3 \cos^2 x - 2 \cos x \\
 &= \cos x (3 \cos^2 x - 2) \\
 &= \cos x (1 - 3 \sin^2 x)
 \end{aligned}$$

N168b

$$(4) \quad y = \sin 2x \cos 3x$$

$$[\text{Sol}] \quad y' = (\sin 2x)' \cos 3x + \sin 2x (\cos 3x)'$$

$$\begin{aligned} &= 2 \cos 2x \cos 3x - 3 \sin 2x \sin 3x \\ &= (\cos 5x + \cos x) + \frac{3}{2} (\cos 5x - \cos x) \\ &= \frac{1}{2} (5 \cos 5x - \cos x) \end{aligned}$$

$$\begin{aligned} \cos \alpha \cos \beta &= \frac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)] \\ \sin \alpha \sin \beta &= -\frac{1}{2} [\cos (\alpha + \beta) - \cos (\alpha - \beta)] \\ \cos (-\theta) &= \cos \theta \end{aligned}$$

Alternative Solution

$$y = \frac{1}{2} (\sin 5x - \sin x)$$

$$y' = \frac{1}{2} (5 \cos 5x - \cos x)$$

$$\begin{aligned} \sin \alpha \cos \beta &= \frac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)] \\ \sin (-\theta) &= -\sin \theta \end{aligned}$$

$$(5) \quad y = \sin 2x \tan x$$

$$[\text{Sol}] \quad y' = (\sin 2x)' \tan x + \sin 2x (\tan x)'$$

$$= 2 \cos 2x \tan x + 2 \sin x \cos x \cdot \frac{1}{\cos^2 x}$$

$$= 2 \cos 2x \tan x + 2 \tan x$$

$$= 2 (\cos 2x + 1) \tan x$$

$$= 2 \cdot 2 \cos^2 x \cdot \frac{\sin x}{\cos x}$$

$$= 4 \sin x \cos x$$

$$= 2 \sin 2x$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = 2 \cos^2 x - 1$$

Alternative Solution 1

$$y' = (\sin 2x)' \tan x + \sin 2x (\tan x)'$$

$$= 2 \cos 2x \cdot \frac{\sin x}{\cos x} + \sin 2x \cdot \frac{1}{\cos^2 x}$$

$$= \frac{\cos 2x \sin 2x + \sin 2x}{\cos^2 x} = \frac{\sin 2x (\cos 2x + 1)}{\cos^2 x}$$

$$= \frac{\sin 2x \cdot 2 \cos^2 x}{\cos^2 x} = 2 \sin 2x$$

Alternative Solution 2

$$y = 2 \sin x \cos x \cdot \frac{\sin x}{\cos x}$$

$$= 2 \sin^2 x$$

$$y' = 2 \cdot 2 \sin x \cdot (\sin x)'$$

$$= 4 \sin x \cos x$$

$$= 2 \sin 2x$$

Differentiation of Trigonometric Functions

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

1. Given that x and y satisfy the following equation, express $\frac{dy}{dx}$ in terms of

$$x = \tan y \quad \left(-\frac{\pi}{2} < y < \frac{\pi}{2} \right)$$

$$\begin{aligned} \text{[Sol]} \quad \frac{dx}{dy} &= \frac{1}{\cos^2 y} \\ &= 1 + \tan^2 y \end{aligned}$$

$$= 1 + x^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

2. Given $f(x) = \frac{\sin^2 x}{x}$ ($x \neq 0$), find $\lim_{x \rightarrow 0} f'(x)$.

$$\begin{aligned} \text{[Sol]} \quad f'(x) &= \frac{(\sin^2 x)' \cdot x - \sin^2 x \cdot (x)'}{x^2} \\ &= \frac{2x \sin x \cdot (\sin x)' - \sin^2 x \cdot 1}{x^2} \\ &= \frac{2x \sin x \cos x - \sin^2 x}{x^2} \end{aligned}$$

$$= \frac{\sin x}{x} \cdot 2 \cos x - \left(\frac{\sin x}{x} \right)^2$$

$$\therefore \lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \cdot 2 \cos x - \left(\frac{\sin x}{x} \right)^2 \right]$$

$$= 1$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

3. Let $\frac{d}{dx}(\sin x \cos x \sqrt{2 - \sin^2 x}) = \frac{a \sin^4 x + b \sin^2 x + c}{\sqrt{2 - \sin^2 x}}$. Find the values of constants a , b and c .

[Sol] $\sin x \cos x \sqrt{2 - \sin^2 x} = \sin x \cos x (2 - \sin^2 x)^{\frac{1}{2}}$

$$\frac{d}{dx}(\sin x \cos x \sqrt{2 - \sin^2 x})$$

$$= (\sin x \cos x)' (2 - \sin^2 x)^{\frac{1}{2}} + \sin x \cos x \left[(2 - \sin^2 x)^{\frac{1}{2}} \right]'$$

$$= [(\sin x)' \cos x + \sin x (\cos x)'] (2 - \sin^2 x)^{\frac{1}{2}}$$

$$+ \sin x \cos x \cdot \frac{1}{2} (2 - \sin^2 x)^{-\frac{1}{2}} \cdot (-2 \sin x)$$

$$= (\cos^2 x - \sin^2 x) \sqrt{2 - \sin^2 x} + \frac{\sin x \cos x}{2 \sqrt{2 - \sin^2 x}} \cdot (-2) \sin x \cdot (\sin x)'$$

$$= \frac{(\cos^2 x - \sin^2 x)(2 - \sin^2 x) - \sin^2 x \cos^2 x}{\sqrt{2 - \sin^2 x}}$$

$$= \frac{(1 - 2 \sin^2 x)(2 - \sin^2 x) - \sin^2 x (1 - \sin^2 x)}{\sqrt{2 - \sin^2 x}}$$

$$= \frac{3 \sin^4 x - 6 \sin^2 x + 2}{\sqrt{2 - \sin^2 x}}$$

$$\therefore a = 3, \quad b = -6, \quad c = 2$$

Alternative Solution

It is possible to solve as follows:

$$\frac{d}{dx}(\sin x \cos x \sqrt{2 - \sin^2 x})$$

$$= (\sin x)' \cos x (2 - \sin^2 x)^{\frac{1}{2}} + \sin x (\cos x)' (2 - \sin^2 x)^{\frac{1}{2}} + \sin x \cos x \left[(2 - \sin^2 x)^{\frac{1}{2}} \right]' \leftarrow$$

Alternatively, $\sin x \cos x = \frac{1}{2} \sin 2x$ can also be used.

Differentiation of
Trigonometric Functions

Name _____

Date / /

Time : to :

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(mistakes) 0	—	1	—	2~

Differentiate the following functions.

(1) $y = (x^2 + 1) \sin x$

➡ N1

$$\begin{aligned}
 \text{[Sol]} \quad y' &= (x^2 + 1)' \sin x + (x^2 + 1)(\sin x)' \\
 &= 2x \sin x + (x^2 + 1) \cos x
 \end{aligned}$$

(2) $y = -\cos 2x + \sin 3x$

➡ N

$$\text{[Sol]} \quad y' = 2 \sin 2x + 3 \cos 3x$$

(3) $y = \tan^3 x$

➡ N

$$\begin{aligned}
 \text{[Sol]} \quad y' &= 3 \tan^2 x \cdot (\tan x)' \\
 &= 3 \tan^2 x \cdot \frac{1}{\cos^2 x} \\
 &= \frac{3 \tan^2 x}{\cos^2 x} \quad \left[= \frac{3 \sin^2 x}{\cos^4 x} \right]
 \end{aligned}$$

N170b

$$(4) \quad y = \sqrt{1 - \cos x}$$

$$[\text{Sol}] \quad y = (1 - \cos x)^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \frac{1}{2} (1 - \cos x)^{-\frac{1}{2}} \cdot (1 - \cos x)' \\ &= \frac{\sin x}{2\sqrt{1 - \cos x}} \end{aligned}$$

$$(5) \quad y = \frac{1}{\sin x}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= -\frac{(\sin x)'}{\sin^2 x} \\ &= -\frac{\cos x}{\sin^2 x} \end{aligned}$$

$$(6) \quad y = \frac{1 - \sin x}{1 + \cos x}$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(1 - \sin x)'(1 + \cos x) - (1 - \sin x)(1 + \cos x)'}{(1 + \cos x)^2} \\ &= \frac{-\cos x(1 + \cos x) + (1 - \sin x)\sin x}{(1 + \cos x)^2} \\ &= \frac{-\cos x + \sin x - (\cos^2 x + \sin^2 x)}{(1 + \cos x)^2} \\ &= \frac{-\cos x - \sin x + 1}{(1 + \cos x)^2} \end{aligned}$$

Differentiation of Logarithmic and Exponential Functions

Name _____

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(minutes) 0	—	—	1	2~

Using the definition of the derivative, differentiate $y = \log_a x$. ($x > 0$)

[Sol] $y' = \lim_{h \rightarrow 0} \frac{\log_a(x+h) - \log_a x}{h}$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \log_a \frac{x+h}{x}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \log_a \left(1 + \frac{h}{x}\right)$$

Let $\frac{h}{x} = k$. As $h \rightarrow 0$, $k \rightarrow 0$; therefore,

$$y' = \lim_{k \rightarrow 0} \frac{1}{kx} \log_a(1+k)$$

$$= \lim_{k \rightarrow 0} \frac{1}{x} \log_a(1+k)^{\frac{1}{k}}$$

$$\log_a M - \log_a N = \log_a \frac{M}{N}$$

$$n \log_a M = \log_a M^n$$

As $k \rightarrow 0$, $(1+k)^{\frac{1}{k}}$ is known to approach a constant value, which is expressed as e .

$$\text{So, } e = \lim_{k \rightarrow 0} (1+k)^{\frac{1}{k}}$$

Using this constant value e ,

$$y' = \frac{1}{x} \log_a e = \frac{\log_e e}{x \log_e a} = \frac{1}{x \log_e a} \quad \dots \textcircled{1}$$

$$\log_a M = \frac{\log M}{\log a}$$

$$\log_a a = 1$$

Notes: All the answers are the same, x

When $a = e$, $\log_e a = \log_e e = 1$; therefore, from $\textcircled{1}$, $y' = \frac{1}{x}$

When examining the value of $(1+k)^{\frac{1}{k}}$ by substituting k with a value which is close to 0, it approaches a constant value as shown on the right. e is an irrational number and $e = 2.7182818\dots$

k	$(1+k)^{\frac{1}{k}}$	k	$(1+k)^{\frac{1}{k}}$
0.1	2.59374...	-0.1	2.867...
0.01	2.70481...	-0.01	2.73...
0.001	2.71692...	-0.001	2.71...
0.0001	2.71814...	-0.0001	2.71...
0.00001	2.71826...	-0.00001	2.71...

The logarithm of x to the base e , $\log_e x$, is called the *natural logarithm*. For differentiation and integration, it is often expressed as $\ln x$, i.e. $\log_e x = \ln x$.

Derivatives of Logarithmic Functions I

$$(\ln x)' = \frac{1}{x}, \quad (\log_a x)' = \frac{1}{x \ln a}$$

Differentiate the following functions.

Ex.

$$y = x \ln x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x)' \ln x + x (\ln x)' \\ &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\ &= \ln x + 1 \end{aligned}$$

$$\begin{aligned} [f(x)g(x)]' &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

$$y = x \log_2 x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x)' \log_2 x + x (\log_2 x)' \\ &= 1 \cdot \log_2 x + x \cdot \frac{1}{x \ln 2} \\ &= \log_2 x + \frac{1}{\ln 2} \end{aligned}$$

$$(1) \quad y = x^2 \ln x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^2)' \ln x + x^2 (\ln x)' \\ &= 2x \ln x + x^2 \cdot \frac{1}{x} \\ &= 2x \ln x + x \\ &= x(2 \ln x + 1) \end{aligned}$$

$$(3) \quad y = x^2 \log_3 x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^2)' \log_3 x + x^2 (\log_3 x)' \\ &= 2x \log_3 x + x^2 \cdot \frac{1}{x \ln 3} \\ &= 2x \log_3 x + \frac{x}{\ln 3} \\ &= \frac{x(2 \log_3 x + 1)}{\ln 3} \end{aligned}$$

$$\log_a M = \frac{\log_b M}{\log_b a}$$

$$(2) \quad y = x^3 \ln x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^3)' \ln x + x^3 (\ln x)' \\ &= 3x^2 \ln x + x^3 \cdot \frac{1}{x} \\ &= 3x^2 \ln x + x^2 \\ &= x^2(3 \ln x + 1) \end{aligned}$$

$$(4) \quad y = x^3 \log_5 x$$

$$\begin{aligned} \text{[Sol]} \quad y' &= (x^3)' \log_5 x + x^3 (\log_5 x)' \\ &= 3x^2 \log_5 x + x^3 \cdot \frac{1}{x \ln 5} \\ &= 3x^2 \log_5 x + \frac{x^2}{\ln 5} \\ &= \frac{x^2(3 \log_5 x + 1)}{\ln 5} \end{aligned}$$

Differentiation of Logarithmic and Exponential Functions

Name _____

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problems 0	1	2	3	4

1. Differentiate the following functions.

Ex.

$$y = \ln(3x-1)$$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(3x-1)'}{3x-1} \\ &= \frac{3}{3x-1} \end{aligned}$$



$$\begin{aligned} [f(g(x))]' &= f'(g(x))g'(x), \\ \text{i.e. } [\ln g(x)]' &= \frac{g'(x)}{g(x)} \end{aligned}$$

(1) $y = \ln(2x+1)$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(2x+1)'}{2x+1} \\ &= \frac{2}{2x+1} \end{aligned}$$

(4) $y = \log_2 3x$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(3x)'}{3x \ln 2} \\ &= \frac{3}{3x \ln 2} \\ &= \frac{1}{x \ln 2} \end{aligned}$$

(2) $y = \ln(x^2-x+1)$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(x^2-x+1)'}{x^2-x+1} \\ &= \frac{2x-1}{x^2-x+1} \end{aligned}$$

(5) $y = \log_5(4x-1)$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(4x-1)'}{(4x-1) \ln 5} \\ &= \frac{4}{(4x-1) \ln 5} \end{aligned}$$

(3) $y = \ln(1-x^2)$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(1-x^2)'}{1-x^2} \\ &= -\frac{2x}{1-x^2} \\ &= -\frac{2x}{(1+x)(1-x)} \end{aligned}$$

(6) $y = \log_3(2-5x)$

$$\begin{aligned} \text{[Sol]} \quad y' &= \frac{(2-5x)'}{(2-5x) \ln 3} \\ &= -\frac{5}{(2-5x) \ln 3} \end{aligned}$$

In N171, the derivative of $y = \log_a x$ when $x > 0$ is determined. Here, consider the derivative of $y = \log_a(-x)$ when $x < 0$ by filling in the following blanks.

$$[1] \quad [\log_a(-x)]' = \frac{(-x)'}{-x \ln a} = \boxed{\frac{1}{x \ln a}}$$

$$[2] \quad [\ln(-x)]' = \frac{(-x)'}{-x} = \boxed{\frac{1}{x}}$$

Answers: $\frac{x}{1}, \frac{d \ln x}{1}$

From the above, combining the conditions when $x > 0$ and $x < 0$, the derivatives of logarithmic functions can be expressed as follows.

Derivatives of Logarithmic Functions II

$$(\ln |x|)' = \frac{1}{x}, \quad (\log_a |x|)' = \frac{1}{x \ln a}$$

2. Differentiate the following functions.

Ex.

$$y = \ln |2x - 3|$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(2x-3)'}{2x-3} \\ &= \frac{2}{2x-3} \end{aligned}$$

$$y = \log_{10} |3x|$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(3x)'}{3x \ln 10} \\ &= \frac{1}{x \ln 10} \end{aligned}$$

$$y = \ln |4x + 1|$$

$$\begin{aligned} y' &= \frac{(4x+1)'}{4x+1} \\ &= \frac{4}{4x+1} \end{aligned}$$

$$y = \ln |x^2 - 3|$$

$$\begin{aligned} y' &= \frac{(x^2-3)'}{x^2-3} \\ &= \frac{2x}{x^2-3} \end{aligned}$$

$$(3) \quad y = \log_3 |5x|$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(5x)'}{5x \ln 3} \\ &= \frac{1}{x \ln 3} \end{aligned}$$

$$(4) \quad y = \log_2 |1 - 2x|$$

$$\begin{aligned} [\text{Sol}] \quad y' &= \frac{(1-2x)'}{(1-2x) \ln 2} \\ &= -\frac{2}{(1-2x) \ln 2} \end{aligned}$$

Differentiation of Logarithmic and Exponential Functions

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Differentiate the following functions.

(1) $y = \ln(\cos x)$

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(\cos x)'}{\cos x} \\
 &= -\frac{\sin x}{\cos x} \\
 &= -\tan x
 \end{aligned}$$

(2) $y = \ln|\ln x|$

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(\ln x)'}{\ln x} \\
 &= \frac{1}{x \ln x}
 \end{aligned}$$

(3) $y = \ln(x + \sqrt{x^2 + 1})$

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(x + \sqrt{x^2 + 1})'}{x + \sqrt{x^2 + 1}} \\
 &= \frac{1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2 + 1}} \\
 &= \frac{\sqrt{x^2 + 1} + x}{(x + \sqrt{x^2 + 1})\sqrt{x^2 + 1}} \\
 &= \frac{1}{\sqrt{x^2 + 1}}
 \end{aligned}$$

$$\begin{aligned}
 (x + \sqrt{x^2 + 1})' &= [x + (x^2 + 1)^{\frac{1}{2}}]' \\
 &= 1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot (x^2)'
 \end{aligned}$$

N173b

(4) $y = x \ln 2x$

[Sol] $y' = (x)' \ln 2x + x (\ln 2x)'$ $\leftarrow [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

$$= 1 \cdot \ln 2x + x \cdot \frac{(2x)'}{2x}$$

$$= \ln 2x + 1$$

(5) $y = \sqrt{x} \ln x$

[Sol] $y' = (\sqrt{x})' \ln x + \sqrt{x} (\ln x)'$

$$= \frac{1}{2} x^{-\frac{1}{2}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x}$$

$$= \frac{\ln x + 2}{2\sqrt{x}} \quad \left[= \frac{\sqrt{x}(\ln x + 2)}{2x} \right]$$

(6) $y = \frac{\ln x}{x}$

[Sol] $y' = \frac{(\ln x)' \cdot x - \ln x \cdot (x)'}{x^2}$ $\leftarrow \left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$$= \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

(7) $y = \frac{\ln x}{\sqrt{x}}$

[Sol] $y' = \frac{(\ln x)' \cdot \sqrt{x} - \ln x \cdot (\sqrt{x})'}{x}$

$$= \frac{\frac{1}{x} \cdot \sqrt{x} - \ln x \cdot \frac{1}{2} x^{-\frac{1}{2}}}{x}$$

$$= \frac{2 - \ln x}{2x\sqrt{x}} \quad \left[= \frac{\sqrt{x}(2 - \ln x)}{2x^{\frac{3}{2}}} \right]$$

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Differentiate the following functions.

Ex. $y = (\ln x)^3$ ← $y = \blacksquare^3$

[Sol] $y' = 3(\ln x)^2 \cdot (\ln x)'$ ← $y' = 3\blacksquare^2 \cdot \blacksquare'$

$$= \frac{3(\ln x)^2}{x}$$

(1) $y = (\ln x)^4$

[Sol] $y' = 4(\ln x)^3 \cdot (\ln x)'$

$$= \frac{4(\ln x)^3}{x}$$

(2) $y = (\ln 3x)^2$

[Sol] $y' = 2\ln 3x \cdot (\ln 3x)'$

$$= 2\ln 3x \cdot \frac{(3x)'}{3x}$$

$$= \frac{2\ln 3x}{x}$$

(3) $y = x(\ln x)^2$

[Sol] $y' = (x)'(\ln x)^2 + x[(\ln x)^2]'$

$$= 1 \cdot (\ln x)^2 + x \cdot 2\ln x \cdot (\ln x)'$$

$$= (\ln x)^2 + 2\ln x$$

$$[= \ln x(\ln x + 2)]$$

Ex.

$$y = \ln \left| \frac{3x-1}{2x+1} \right|$$

$$[\text{Sol}] \quad y = \ln |3x-1| - \ln |2x+1|$$

$$\ln \frac{M}{N} = \ln M - \ln N$$

$$\begin{aligned} y' &= \frac{(3x-1)'}{3x-1} - \frac{(2x+1)'}{2x+1} \\ &= \frac{3}{3x-1} - \frac{2}{2x+1} \\ &= \frac{5}{(3x-1)(2x+1)} \end{aligned}$$

$$(4) \quad y = \ln \left| \frac{x+1}{x-1} \right|$$

$$[\text{Sol}] \quad y = \ln |x+1| - \ln |x-1|$$

$$\begin{aligned} y' &= \frac{(x+1)'}{x+1} - \frac{(x-1)'}{x-1} \\ &= \frac{1}{x+1} - \frac{1}{x-1} \\ &= -\frac{2}{(x+1)(x-1)} \end{aligned}$$

$$(5) \quad y = \ln \frac{1+\sin x}{1-\sin x}$$

$$[\text{Sol}] \quad y = \ln(1+\sin x) - \ln(1-\sin x)$$

$$\begin{aligned} y' &= \frac{(1+\sin x)'}{1+\sin x} - \frac{(1-\sin x)'}{1-\sin x} \\ &= \frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x} \\ &= \frac{2\cos x}{(1+\sin x)(1-\sin x)} \\ &= \frac{2\cos x}{1-\sin^2 x} \\ &= \frac{2}{\cos x} \end{aligned}$$

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

Differentiation of Logarithmic and Exponential Functions

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Ex.Differentiate $y = \frac{(x-2)^2}{x+1}$.

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y| = \ln \left| \frac{(x-2)^2}{x+1} \right|$$

$$= 2\ln|x-2| - \ln|x+1|$$

$$\ln \frac{M}{N} = \ln M - \ln N$$

$$\ln M^n = n \ln M$$

Differentiating both sides with respect to x ,

$$\frac{y'}{y} = \frac{2}{x-2} - \frac{1}{x+1}$$

$$= \frac{x+4}{(x-2)(x+1)}$$

$$(\ln|y|)' = \frac{y'}{y}$$

$$\therefore y' = \frac{x+4}{(x-2)(x+1)} \cdot \frac{(x-2)^2}{x+1} = \frac{(x+4)(x-2)}{(x+1)^2}$$

As shown above, differentiation by taking the natural logarithm of both sides called *logarithmic differentiation*.

Differentiate the following functions using logarithmic differentiation.

(1) $y = \frac{x-1}{(x+3)^2}$

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y| = \ln \left| \frac{x-1}{(x+3)^2} \right|$$

$$= \ln|x-1| - 2\ln|x+3|$$

Differentiating both sides with respect to x ,

$$\frac{y'}{y} = \frac{1}{x-1} - \frac{2}{x+3}$$

$$= -\frac{x-5}{(x-1)(x+3)}$$

$$\therefore y' = -\frac{x-5}{(x-1)(x+3)} \cdot \frac{x-1}{(x+3)^2} = -\frac{x-5}{(x+3)^3}$$

$$|a^2| = |a|^2$$

N175b

$$(2) \quad y = \frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\begin{aligned} \ln|y| &= \ln \left| \frac{(x+1)^3}{(x-2)^2(x+3)^4} \right| \\ &= 3\ln|x+1| - (2\ln|x-2| + 4\ln|x+3|) \end{aligned}$$

$\ln MN = \ln M + \ln N$

Differentiating both sides with respect to x ,

$$\begin{aligned} \frac{y'}{y} &= \frac{3}{x+1} - \left(\frac{2}{x-2} + \frac{4}{x+3} \right) \\ &= -\frac{3x^2+x+16}{(x+1)(x-2)(x+3)} \\ \therefore y' &= -\frac{3x^2+x+16}{(x+1)(x-2)(x+3)} \cdot \frac{(x+1)^3}{(x-2)^2(x+3)^4} \\ &= -\frac{(3x^2+x+16)(x+1)^2}{(x-2)^3(x+3)^5} \end{aligned}$$

$$(3) \quad y = \frac{x}{\sqrt{2x+1}}$$

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\begin{aligned} \ln|y| &= \ln \left| \frac{x}{\sqrt{2x+1}} \right| \\ &= \ln|x| - \frac{1}{2}\ln|2x+1| \end{aligned}$$

$\ln|\sqrt{2x+1}| = \ln|2x+1|^{\frac{1}{2}} = \frac{1}{2}\ln|2x+1|$

Differentiating both sides with respect to x ,

$$\begin{aligned} \frac{y'}{y} &= \frac{1}{x} - \frac{(2x+1)'}{2(2x+1)} \\ &= \frac{1}{x} - \frac{1}{2x+1} \\ &= \frac{x+1}{x(2x+1)} \\ \therefore y' &= \frac{x+1}{x(2x+1)} \cdot \frac{x}{\sqrt{2x+1}} = \frac{x+1}{(2x+1)\sqrt{2x+1}} \quad \left[= \frac{(x+1)\sqrt{2x+1}}{(2x+1)^2} \right] \end{aligned}$$

$$|ab| = |a||b|$$

Differentiation of Logarithmic and Exponential Functions

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1. Differentiate the following functions using logarithmic differentiation.

(1) $y = x^x \quad (x > 0)$

[Sol] Since $x > 0, y > 0$

Taking the natural logarithm of both sides,

$$\ln y = \ln x^x$$

$$= x \ln x$$

$$\ln M^n = n \ln M$$

Since $x > 0$ and $y > 0$, the absolute value symbol does not have to be included when taking the logarithm.Differentiating both sides with respect to x ,

$$\frac{y'}{y} = (x)' \ln x + x (\ln x)'$$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\therefore y' = (\ln x + 1) \cdot x^x = x^x (\ln x + 1)$$

(2) $y = x^{\ln x}$

[Sol] Since the antilogarithm should be a positive number, $x > 0$

L21

Since $x > 0, y > 0$

Taking the natural logarithm of both sides,

$$\ln y = \ln x^{\ln x}$$

$$= \ln x \cdot \ln x$$

$$= (\ln x)^2$$

Differentiating both sides with respect to x ,

$$\frac{y'}{y} = 2 \ln x \cdot (\ln x)'$$

$$= \frac{2 \ln x}{x}$$

$$\therefore y' = \frac{2 \ln x}{x} \cdot x^{\ln x} = 2x^{\ln x - 1} \ln x$$

Differentiation of Logarithmic and Exponential Functions

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Differentiate $y = a^x$. ($a > 0$, $a \neq 1$)[Sol] Since $a > 0$, $y > 0$

Taking the natural logarithm of both sides,

$$\ln y = \ln a^x$$

$$= x \ln a$$

$$\ln M^n = n \ln M$$

Differentiating both sides with respect to x ,

$$\frac{y'}{y} = \ln a$$

$$\therefore y' = y \ln a, \text{ i.e. } (a^x)' = a^x \ln a \quad \dots \textcircled{1}$$

Answers: All the answers are the same, $\ln a$ When $a = e$, $\ln e = \log_e e = 1$; therefore, from $\textcircled{1}$, $(e^x)' = e^x$

Derivatives of Exponential Functions

$$(e^x)' = e^x, \quad (a^x)' = a^x \ln a$$

Differentiate the following functions. ($a > 0$, $a \neq 1$)

Ex.

$$y = e^{2x}$$

$$[f(g(x))]' = f'(g(x))g'(x)$$

$$y = 2^{3x}$$

$$[\text{Sol}] y' = e^{2x} \cdot (2x)'$$

$$= 2e^{2x}$$

$$[\text{Sol}] y' = 2^{3x} \ln 2 \cdot (3x)'$$

$$= 3 \cdot 2^{3x} \ln 2$$

$$(1) y = e^{4x}$$

$$[\text{Sol}] y' = e^{4x} \cdot (4x)'$$

$$= 4e^{4x}$$

$$(2) y = 3^{2x}$$

$$[\text{Sol}] y' = 3^{2x} \ln 3 \cdot (2x)'$$

$$= 2 \cdot 3^{2x} \ln 3$$

(3) $y = 5^{-x}$

$$\begin{aligned} \text{[Sol]} \quad y' &= 5^{-x} \ln 5 \cdot (-x)' \\ &= -5^{-x} \ln 5 \end{aligned}$$

(7) $y = e^{\frac{1}{x}}$

$$\begin{aligned} \text{[Sol]} \quad y' &= e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)' \\ &= -\frac{e^{\frac{1}{x}}}{x^2} \end{aligned}$$

(4) $y = a^{3x}$

$$\begin{aligned} \text{[Sol]} \quad y' &= a^{3x} \ln a \cdot (3x)' \\ &= 3a^{3x} \ln a \end{aligned}$$

(8) $y = e^{\sqrt{x}}$

$$\begin{aligned} \text{[Sol]} \quad y' &= e^{\sqrt{x}} \cdot (\sqrt{x})' \\ &= e^{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} \\ &= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \end{aligned}$$

(5) $y = e^{x^2}$

$$\begin{aligned} \text{[Sol]} \quad y' &= e^{x^2} \cdot (x^2)' \\ &= 2xe^{x^2} \end{aligned}$$

(9) $y = e^{\sin x}$

$$\begin{aligned} \text{[Sol]} \quad y' &= e^{\sin x} \cdot (\sin x)' \\ &= e^{\sin x} \cos x \end{aligned}$$

(6) $y = e^{-x^3}$

$$\begin{aligned} \text{[Sol]} \quad y' &= e^{-x^3} \cdot (-x^3)' \\ &= -3x^2 e^{-x^3} \end{aligned}$$

(10) $y = 2^{\ln x}$

$$\begin{aligned} \text{[Sol]} \quad y' &= 2^{\ln x} \ln 2 \cdot (\ln x)' \\ &= \frac{2^{\ln x} \ln 2}{x} \end{aligned}$$

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Differentiate the following functions. ($a > 0$, $a \neq 1$)**Ex.**

$y = xe^{2x}$

[Sol] $y' = (x)'e^{2x} + x(e^{2x})'$

$$\begin{aligned} [f(x)g(x)]' \\ = f'(x)g(x) + f(x)g'(x) \end{aligned}$$

$= 1 \cdot e^{2x} + 2xe^{2x}$

$= (2x+1)e^{2x}$

(1) $y = (x+3)e^x$

(4) $y = xa^x$

[Sol] $y' = (x+3)'e^x + (x+3)(e^x)'$

[Sol] $y' = (x)'a^x + x(a^x)'$

$= 1 \cdot e^x + (x+3)e^x$

$= 1 \cdot a^x + xa^x \ln a$

$= (x+4)e^x$

$= a^x(1+x \ln a)$

(2) $y = xe^{-x^2}$

(5) $y = e^x \cos x$

[Sol] $y' = (x)'e^{-x^2} + x(e^{-x^2})'$

[Sol] $y' = (e^x)' \cos x + e^x(\cos x)'$

$= 1 \cdot e^{-x^2} - 2x^2 e^{-x^2}$

$= e^x \cos x - e^x \sin x$

$= (1-2x^2)e^{-x^2}$

$= e^x(\cos x - \sin x)$

(3) $y = \sqrt{x}e^x$

(6) $y = e^x \ln x$

[Sol] $y' = (\sqrt{x})'e^x + \sqrt{x}(e^x)'$

[Sol] $y' = (e^x)' \ln x + e^x(\ln x)'$

$= \frac{1}{2}x^{-\frac{1}{2}} \cdot e^x + \sqrt{x}e^x$

$= e^x \ln x + e^x \cdot \frac{1}{x}$

$= \frac{(2x+1)e^x}{2\sqrt{x}}$

$= e^x \left(\ln x + \frac{1}{x} \right)$

$$\left[= \frac{(2x+1)\sqrt{x}e^x}{2x} \right]$$

$$\left[= \frac{e^x(x \ln x + 1)}{x} \right]$$

$$(7) \quad y = \frac{e^x}{x}$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \frac{(e^x)' \cdot x - e^x \cdot (x)'}{x^2} \quad \leftarrow \quad \boxed{\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}} \\
 &= \frac{x e^x - e^x \cdot 1}{x^2} \\
 &= \frac{(x-1)e^x}{x^2}
 \end{aligned}$$

$$(8) \quad y = \frac{e^x - 1}{e^x + 1}$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \frac{(e^x - 1)'(e^x + 1) - (e^x - 1)(e^x + 1)'}{(e^x + 1)^2} \\
 &= \frac{e^x(e^x + 1) - (e^x - 1)e^x}{(e^x + 1)^2} \\
 &= \frac{2e^x}{(e^x + 1)^2}
 \end{aligned}$$

$$(9) \quad y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \frac{(e^x - e^{-x})'(e^x + e^{-x}) - (e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{4}{(e^x + e^{-x})^2} \quad \left[= \frac{4e^{2x}}{(e^{2x} + 1)^2} \right]
 \end{aligned}$$

Alternative Solution

$$\begin{aligned}
 y &= \frac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})} = \frac{e^{2x} - 1}{e^{2x} + 1}, \quad y' = \frac{(e^{2x} - 1)'(e^{2x} + 1) - (e^{2x} - 1)(e^{2x} + 1)'}{(e^{2x} + 1)^2} \\
 &= \frac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1) \cdot 2e^{2x}}{(e^{2x} + 1)^2} = \frac{4e^{2x}}{(e^{2x} + 1)^2}
 \end{aligned}$$

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1. Differentiate the following functions.

(1) $y = \ln[e^x(1-x)]$

$$\begin{aligned}
 \text{[Sol]} \quad y' &= \frac{[e^x(1-x)]'}{e^x(1-x)} \\
 &= \frac{(e^x)'(1-x) + e^x(1-x)'}{e^x(1-x)} \\
 &= \frac{e^x(1-x) - e^x}{e^x(1-x)} \\
 &= -\frac{x}{1-x}
 \end{aligned}$$

(2) $y = \ln \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}$

[Sol] $y = \ln(\sqrt{1+e^x}-1) - \ln(\sqrt{1+e^x}+1)$

$$\begin{aligned}
 y' &= \frac{(\sqrt{1+e^x}-1)'}{\sqrt{1+e^x}-1} - \frac{(\sqrt{1+e^x}+1)'}{\sqrt{1+e^x}+1} \\
 &= \frac{\frac{1}{2}(1+e^x)^{-\frac{1}{2}} \cdot e^x}{\sqrt{1+e^x}-1} - \frac{\frac{1}{2}(1+e^x)^{-\frac{1}{2}} \cdot e^x}{\sqrt{1+e^x}+1} \\
 &= \frac{e^x}{2(\sqrt{1+e^x}-1)\sqrt{1+e^x}} - \frac{e^x}{2(\sqrt{1+e^x}+1)\sqrt{1+e^x}} \\
 &= \frac{e^x(\sqrt{1+e^x}+1) - e^x(\sqrt{1+e^x}-1)}{2(\sqrt{1+e^x}-1)(\sqrt{1+e^x}+1)\sqrt{1+e^x}} \\
 &= \frac{2e^x}{2e^x\sqrt{1+e^x}} \\
 &= \frac{1}{\sqrt{1+e^x}}
 \end{aligned}$$

$$\begin{aligned}
 (\sqrt{1+e^x})' &= [(1+e^x)^{\frac{1}{2}}]' \\
 &= \frac{1}{2}(1+e^x)^{-\frac{1}{2}} \cdot (1+e^x)'
 \end{aligned}$$

2. Find the values of a and b that satisfy the following equation.

$$\frac{d}{dx}[x\sqrt{x^2+3} + a\ln(x+\sqrt{x^2+3})] = b\sqrt{x^2+3}$$

[Sol] $\frac{d}{dx}[x\sqrt{x^2+3} + a\ln(x+\sqrt{x^2+3})]$

$$= 1 \cdot \sqrt{x^2+3} + x \cdot \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot 2x + a \cdot \frac{(x+\sqrt{x^2+3})'}{x+\sqrt{x^2+3}}$$

$$= \sqrt{x^2+3} + \frac{x^2}{\sqrt{x^2+3}} + \frac{a\left[1 + \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot 2x\right]}{x+\sqrt{x^2+3}}$$

$$= \frac{2x^2+3}{\sqrt{x^2+3}} + \frac{a(\sqrt{x^2+3}+x)}{(x+\sqrt{x^2+3})\sqrt{x^2+3}}$$

$$= \frac{2x^2+a+3}{\sqrt{x^2+3}}$$

$$\therefore \frac{2x^2+a+3}{\sqrt{x^2+3}} = b\sqrt{x^2+3}$$

$$(2-b)x^2 + (a-3b+3) = 0 \quad \leftarrow \text{Identity for } x \text{ (J185)}$$

$$\therefore 2-b=0, \quad a-3b+3=0$$

$$\therefore a=3, \quad b=2$$

3. Let $f(x) = (ax^2 + bx + c)e^{-x}$ for constants a , b and c .

Given that $f'(x) = f(x) + xe^{-x}$ is satisfied for all real numbers x , find the values of a , b and c .

[Sol] $f'(x) = (ax^2 + bx + c)'e^{-x} + (ax^2 + bx + c)(e^{-x})'$

$$= (2ax + b)e^{-x} - (ax^2 + bx + c)e^{-x}$$

$$= [-ax^2 + (2a-b)x + (b-c)]e^{-x} \quad \dots \textcircled{1}$$

$$f(x) + xe^{-x} = (ax^2 + bx + c)e^{-x} + xe^{-x}$$

$$= [ax^2 + (b+1)x + c]e^{-x} \quad \dots \textcircled{2}$$

From $\textcircled{1}$, $\textcircled{2}$ and $e^{-x} \neq 0$,

$$-a = a, \quad 2a - b = b + 1, \quad b - c = c$$

$$\therefore a = 0, \quad b = -\frac{1}{2}, \quad c = -\frac{1}{4}$$

Coefficient Comparison
Method (J181)

Differentiation of Logarithmic and Exponential Functions

Name _____

Date / /

Time : to :

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1. Differentiate the following functions.

(1) $y = x(\ln x - 1)$

➡ N1

$$\begin{aligned}
 [\text{Sol}] \quad y' &= (x)'(\ln x - 1) + x(\ln x - 1)' \\
 &= 1 \cdot (\ln x - 1) + x \cdot \frac{1}{x} \\
 &= \ln x
 \end{aligned}$$

(2) $y = \ln(2x + 3)$

➡ N1

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(2x + 3)'}{2x + 3} \\
 &= \frac{2}{2x + 3}
 \end{aligned}$$

(3) $y = \log_3 |x^2 - 6|$

➡ N1

$$\begin{aligned}
 [\text{Sol}] \quad y' &= \frac{(x^2 - 6)'}{(x^2 - 6)\ln 3} \\
 &= \frac{2x}{(x^2 - 6)\ln 3}
 \end{aligned}$$

(4) $y = (\ln x)^2$

➡ N1

$$\begin{aligned}
 [\text{Sol}] \quad y' &= 2\ln x \cdot (\ln x)' \\
 &= \frac{2\ln x}{x}
 \end{aligned}$$

2. Differentiate the following function using logarithmic differentiation.

$$y = \frac{\sqrt{x+2}}{x+1}$$

➡ N175

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\begin{aligned}\ln|y| &= \ln \left| \frac{\sqrt{x+2}}{x+1} \right| \\ &= \frac{1}{2} \ln|x+2| - \ln|x+1|\end{aligned}$$

Differentiating both sides with respect to x ,

$$\begin{aligned}\frac{y'}{y} &= \frac{1}{2(x+2)} - \frac{1}{x+1} \\ &= -\frac{x+3}{2(x+2)(x+1)} \\ \therefore y' &= -\frac{x+3}{2(x+2)(x+1)} \cdot \frac{\sqrt{x+2}}{x+1} = -\frac{(x+3)\sqrt{x+2}}{2(x+2)(x+1)^2} \\ &= -\frac{x+3}{2(x+1)^2\sqrt{x+2}}\end{aligned}$$

3. Differentiate the following functions.

(1) $y = e^{3x} + 3^{-x}$

➡ N177

[Sol] $y' = e^{3x} \cdot (3x)' + 3^{-x} \ln 3 \cdot (-x)'$
 $= 3e^{3x} - 3^{-x} \ln 3$

(2) $y = (x-2)e^x$

➡ N178

[Sol] $y' = (x-2)'e^x + (x-2)(e^x)'$
 $= 1 \cdot e^x + (x-2)e^x$
 $= (x-1)e^x$

Differentiation of Various Functions
and Higher Order Derivatives

Name _____

Date / /

Time : to :

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Given that y is a function of x , express $\frac{dy}{dx}$ in terms of x and y for the following equations.

Ex.

$$x^2 + y^2 = 4$$

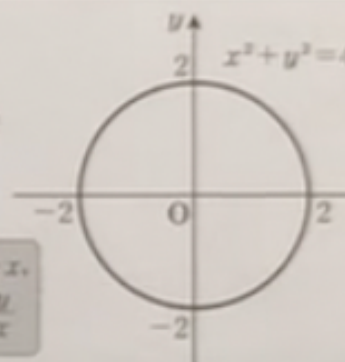
[Sol] Differentiating both sides with respect to x ,

$$2x + 2y \cdot \frac{dy}{dx} = 0 \quad \leftarrow$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Differentiating y^2 with respect to x ,

$$\frac{d}{dx} y^2 = \frac{d}{dy} y^2 \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$$

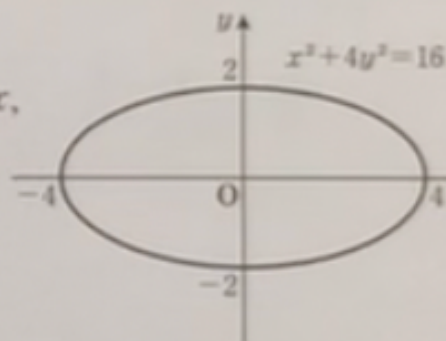


(1) $x^2 + 4y^2 = 16$

[Sol] Differentiating both sides with respect to x ,

$$2x + 8y \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{4y}$$

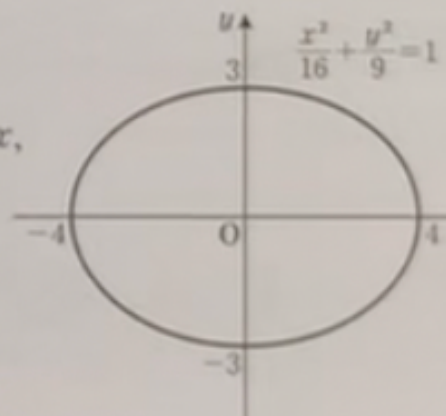


(2) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

[Sol] Differentiating both sides with respect to x ,

$$\frac{x}{8} + \frac{2y}{9} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{9x}{16y}$$

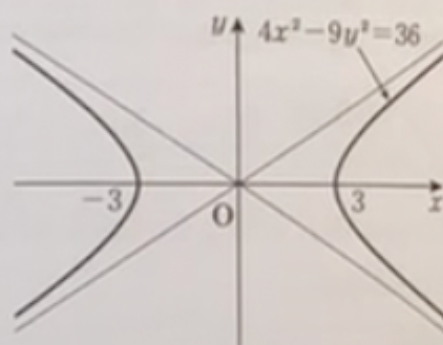


$$(3) \quad 4x^2 - 9y^2 = 36$$

[Sol] Differentiating both sides with respect to x ,

$$8x - 18y \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{4x}{9y}$$

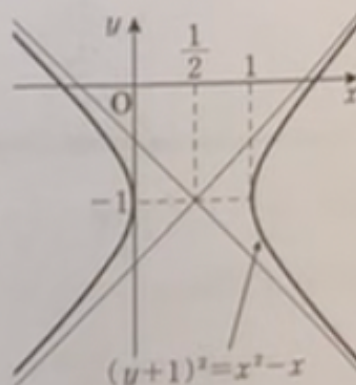


$$(4) \quad (y+1)^2 = x^2 - x$$

[Sol] Differentiating both sides with respect to x ,

$$2(y+1) \cdot \frac{dy}{dx} = 2x - 1$$

$$\therefore \frac{dy}{dx} = \frac{2x-1}{2(y+1)}$$



The derivatives of these types of functions can also be found by solving the equations for y . In the case of **Ex. 3**, the answer can be found as follows.

[Sol] Solving the equation for y , $y = \pm\sqrt{4-x^2}$

(i) When $y = \sqrt{4-x^2}$,

$$\frac{dy}{dx} = (\sqrt{4-x^2})' = \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{4-x^2}} = -\frac{x}{y}$$

(ii) When $y = -\sqrt{4-x^2}$,

$$\frac{dy}{dx} = (-\sqrt{4-x^2})' = -\frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{x}{\sqrt{4-x^2}} = -\frac{x}{y}$$

From (i) and (ii), $\frac{dy}{dx} = -\frac{x}{y}$

Differentiation of Various Functions and Higher Order Derivatives

Name _____

Date / /

Time : to :

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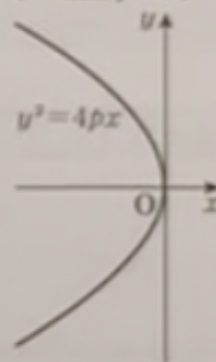
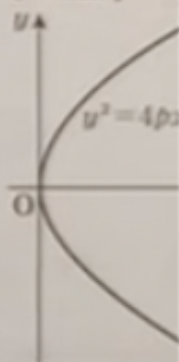
Given that y is a function of x , express $\frac{dy}{dx}$ in terms of x and y for the following equations.

(1) $y^2 = 4px$ (p is a constant)

[Sol] Differentiating both sides with respect to x ,

$$2y \cdot \frac{dy}{dx} = 4p$$

$$\therefore \frac{dy}{dx} = \frac{2p}{y}$$

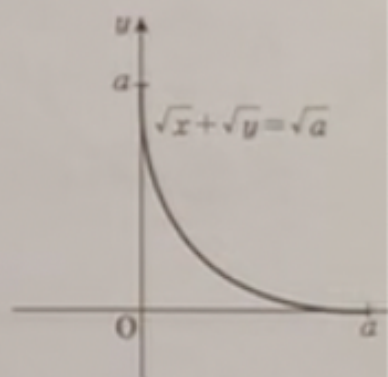
[When $p < 0$][When $p > 0$]

(2) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (a is a constant)

[Sol] Differentiating both sides with respect to x ,

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$



(3) $xy = 1$

[Sol] Differentiating both sides with respect to x ,

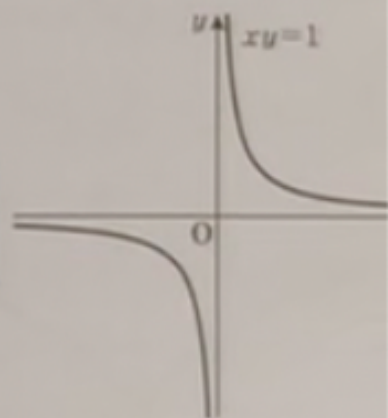
$$y + x \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$(x)'y + x(y)' = 0$$

Since y is not a constant,

$(xy)' = y$ is not true.



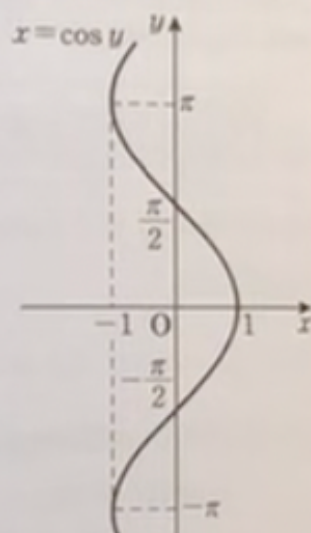
N182b

(4) $x = \cos y$

[Sol] Differentiating both sides with respect to x ,

$$1 = -\sin y \cdot \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sin y}$$



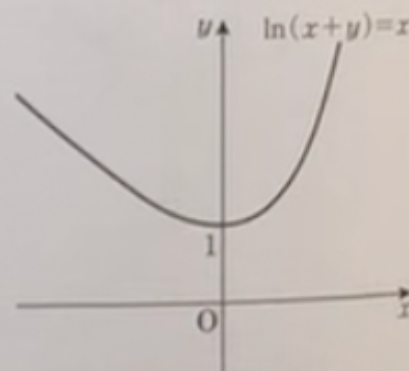
(5) $\ln(x+y) = x$

[Sol] Differentiating both sides with respect to x ,

$$\frac{(x+y)'}{x+y} = 1$$

$$1 + \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = x + y - 1$$



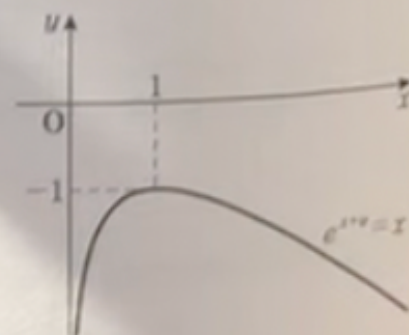
(6) $e^{x+y} = x$

[Sol] Differentiating both sides with respect to x ,

$$e^{x+y} \cdot (x+y)' = 1$$

$$e^{x+y} \left(1 + \frac{dy}{dx} \right) = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^{x+y}} - 1 \quad \left[= \frac{1 - e^{x+y}}{e^{x+y}} \right]$$



Differentiation of Various Functions and Higher Order Derivatives

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Given that y is a function of x and can be expressed as $x=f(t)$ and $y=g(t)$ using the parameter t ,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}}$$



Chain Rule I (N151)

Differentiation Formula for Inverse Functions (N155)

Therefore, the following is true.

Derivative of Functions Represented by a Parameter

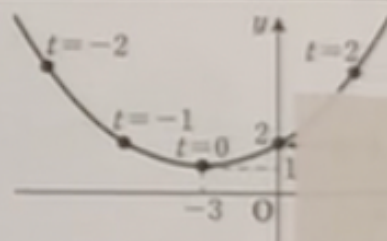
When $x=f(t)$ and $y=g(t)$, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$

Given the following functions represented by parameter t , express $\frac{dy}{dx}$ in terms of t .

Ex. $x=3(t-1)$, $y=t^2+1$

[Sol] $\frac{dx}{dt}=3$, $\frac{dy}{dt}=2t$

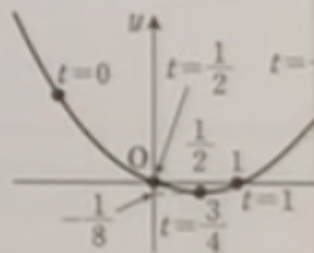
$$\therefore \frac{dy}{dx} = \frac{2}{3}t$$

Differentiating each equation with respect to t 

(1) $x=2t-1$, $y=2t^2-3t+1$

[Sol] $\frac{dx}{dt}=2$, $\frac{dy}{dt}=4t-3$

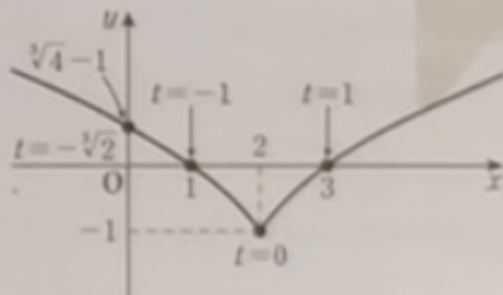
$$\therefore \frac{dy}{dx} = \frac{4t-3}{2}$$



(2) $x=t^3+2$, $y=t^2-1$

[Sol] $\frac{dx}{dt}=3t^2$, $\frac{dy}{dt}=2t$

$$\therefore \frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$$

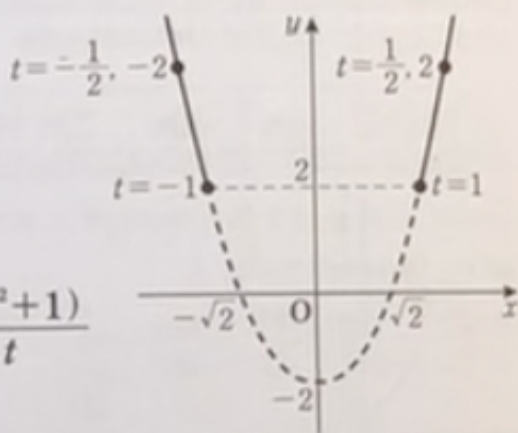


N183b

$$(3) \quad x = t + \frac{1}{t}, \quad y = t^2 + \frac{1}{t^2}$$

$$[\text{Sol}] \quad \frac{dx}{dt} = 1 - \frac{1}{t^2}, \quad \frac{dy}{dt} = 2t - \frac{2}{t^3}$$

$$\therefore \frac{dy}{dx} = \frac{2t - \frac{2}{t^3}}{1 - \frac{1}{t^2}} = \frac{2(t^4 - 1)}{t(t^2 - 1)} = \frac{2(t^2 + 1)}{t}$$

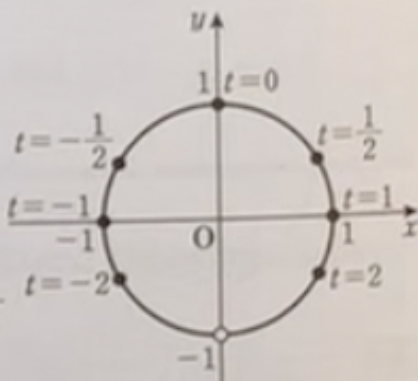


$$(4) \quad x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}$$

$$[\text{Sol}] \quad \frac{dx}{dt} = \frac{2(1+t^2) - 2t \cdot 2t}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{-2t(1+t^2) - (1-t^2) \cdot 2t}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{4t}{(1+t^2)^2}}{\frac{2(1-t^2)}{(1+t^2)^2}} = -\frac{2t}{1-t^2} \quad \left[= -\frac{2t}{(1+t)(1-t)} \right]$$



The derivatives of these types of functions can also be found by solving the equations for y . In the case of **Ex.**, the answer can be found as follows.

$$[\text{Sol}] \quad \text{Since } x = 3(t-1), \quad t = \frac{x+3}{3}$$

$$\text{Substituting into } y = t^2 + 1, \quad y = \left(\frac{x+3}{3}\right)^2 + 1 = \frac{1}{9}(x^2 + 6x + 18)$$

Differentiating both sides with respect to x ,

$$\frac{dy}{dx} = \frac{1}{9}(2x+6) = \frac{2}{9}(x+3)$$

$$\text{Since } x = 3(t-1),$$

$$\frac{dy}{dx} = \frac{2}{9}[3(t-1)+3] = \frac{2}{3}t$$

Differentiation of Various Functions
and Higher Order Derivatives

Name _____

Date / /

Time : to :

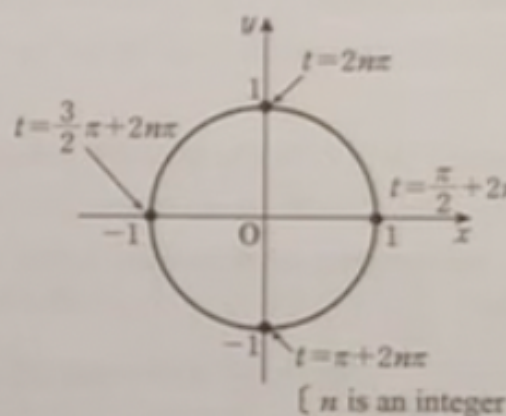
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Given the following functions represented by parameter t , express $\frac{dy}{dx}$ in terms of t .

(1) $x = \sin t$, $y = \cos t$

[Sol] $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = -\sin t$

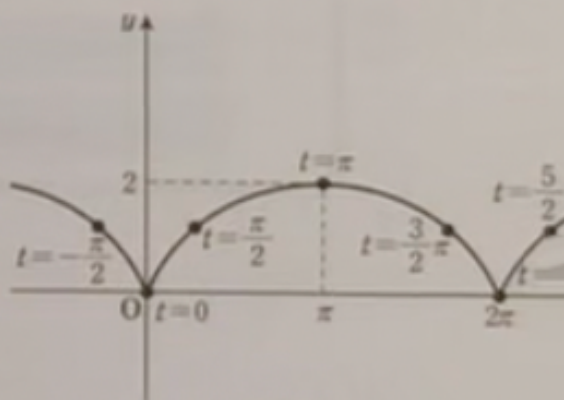
$$\therefore \frac{dy}{dx} = -\frac{\sin t}{\cos t} \quad [= -\tan t]$$



(2) $x = t - \sin t$, $y = 1 - \cos t$

[Sol] $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$

$$\therefore \frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$$

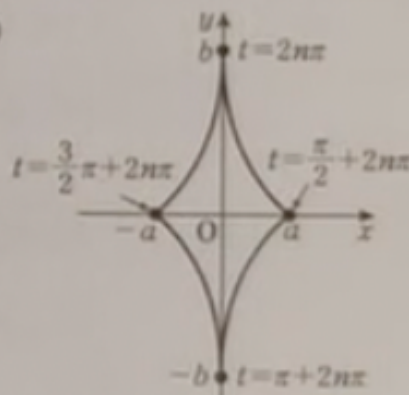


(3) $x = a \sin^3 t$, $y = b \cos^3 t$ (a and b are constants)

[Sol] $\frac{dx}{dt} = 3a \sin^2 t \cdot (\sin t)' = 3a \sin^2 t \cos t$

$$\frac{dy}{dt} = 3b \cos^2 t \cdot (\cos t)' = -3b \cos^2 t \sin t$$

$$\therefore \frac{dy}{dx} = \frac{-3b \cos^2 t \sin t}{3a \sin^2 t \cos t} = -\frac{b \cos t}{a \sin t}$$



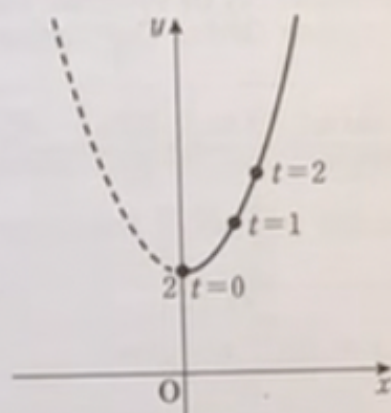
(When $a > 0$ and $b > 0$ (n is an integer))

N184b

(4) $x = \sqrt{t}$, $y = t + 2$

[Sol] $\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$, $\frac{dy}{dt} = 1$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}$$

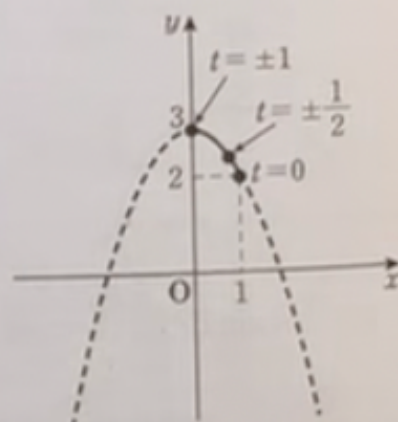


(5) $x = \sqrt{1-t^2}$, $y = t^2 + 2$

[Sol] $\frac{dx}{dt} = \frac{1}{2}(1-t^2)^{-\frac{1}{2}} \cdot (-2t) = -\frac{t}{\sqrt{1-t^2}}$

$$\frac{dy}{dt} = 2t$$

$$\therefore \frac{dy}{dx} = \frac{2t}{-\frac{t}{\sqrt{1-t^2}}} = -2\sqrt{1-t^2}$$

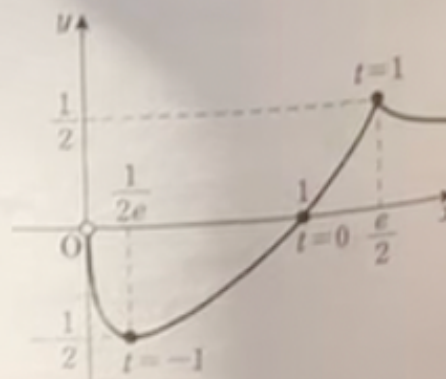


(6) $x = \frac{e^t}{1+t^2}$, $y = \frac{t}{1+t^2}$

$$\begin{aligned} \text{[Sol]} \quad \frac{dx}{dt} &= \frac{(e^t)'(1+t^2) - e^t(1+t^2)'}{(1+t^2)^2} = \frac{e^t(1+t^2) - e^t \cdot 2t}{(1+t^2)^2} = \frac{(1-2t+t^2)e^t}{(1+t^2)^2} \\ &= \frac{(1-t)^2 e^t}{(1+t^2)^2} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{(t)'(1+t^2) - t(1+t^2)'}{(1+t^2)^2} = \frac{1 \cdot (1+t^2) - t \cdot 2t}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2} \\ &= \frac{(1+t)(1-t)}{(1+t^2)^2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{(1+t)(1-t)}{(1+t^2)^2}}{\frac{(1-t)^2 e^t}{(1+t^2)^2}} \\ &= \frac{1+t}{(1-t)e^t} \left[= -\frac{t+1}{(t-1)e^t} \right] \end{aligned}$$



Differentiation of Various Functions
and Higher Order Derivatives

Name _____

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Time : to :

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The function $[f'(x)]'$ which is derived by differentiating the derivative $f'(x)$ of $y=f(x)$ is called the **second order derivative** of $f(x)$, and is expressed as y'' or $f''(x)$. The derivative of the second order derivative $f''(x)$ is called the **third order derivative** of $f(x)$ and is expressed as y''' or $f'''(x)$. Likewise, $f'(x)$ is often called the **first order derivative** of $f(x)$.

Generally, the function found by differentiating the function $y=f(x)$ for n times is called the **n^{th} order derivative** of $f(x)$ and is expressed as $y^{(n)}$ or $f^{(n)}(x)$.

The derivatives of second order and above are called **higher order derivatives**.

Find the second and third order derivatives of the following functions.

Ex. $y = x^4 + 2x^3 - x + 1$

[Sol] $y' = 4x^3 + 6x^2 - 1$

$\therefore y'' = 12x^2 + 12x$

$y''' = 24x + 12$

Differentiating y' Differentiating y''

(1) $y = \frac{1}{2}x^3 - x^2 + 4x$

[Sol] $y' = \frac{3}{2}x^2 - 2x + 4$

$\therefore y'' = 3x - 2$

$y''' = 3$

(2) $y = x - \frac{1}{x}$

[Sol] $y' = 1 + \frac{1}{x^2}$

$\therefore y'' = -\frac{2}{x^3}$

$y''' = \frac{6}{x^4}$

N185b

(3) $y = \sin x$

[Sol] $y' = \cos x$

$$\therefore y'' = -\sin x$$

$$y''' = -\cos x$$

(4) $y = \ln x$

[Sol] $y' = \frac{1}{x}$

$$\therefore y'' = -\frac{1}{x^2}$$

$$y''' = \frac{2}{x^3}$$

(5) $y = e^x$

[Sol] $y' = e^x$

$$\therefore y'' = e^x$$

$$y''' = e^x$$

(6) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

[Sol] $y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

$$\therefore y'' = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}} = -\frac{1}{4x\sqrt{x}} + \frac{3}{4x^2\sqrt{x}} \left[= -\frac{x-3}{4x^2\sqrt{x}} \right]$$

$$y''' = \frac{3}{8}x^{-\frac{5}{2}} - \frac{15}{8}x^{-\frac{7}{2}} = \frac{3}{8x^2\sqrt{x}} - \frac{15}{8x^3\sqrt{x}} \left[= \frac{3(x-5)}{8x^3\sqrt{x}} \right]$$

Differentiation of Various Functions
and Higher Order Derivatives

Name _____

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1. Find the second order derivatives of the following functions.

(1) $y = x \sin x$

[Sol] $y' = (x)' \sin x + x(\sin x)'$

$$= 1 \cdot \sin x + x \cos x$$

$$= \sin x + x \cos x$$

$$\therefore y'' = \cos x + [(x)' \cos x + x(\cos x)']$$

$$= \cos x + (1 \cdot \cos x - x \sin x)$$

$$= 2 \cos x - x \sin x$$

(2) $y = x^2 \ln x$

[Sol] $y' = (x^2)' \ln x + x^2(\ln x)'$

$$= 2x \ln x + x^2 \cdot \frac{1}{x}$$

$$= x(2 \ln x + 1)$$

$$\therefore y'' = (x)'(2 \ln x + 1) + x(2 \ln x + 1)'$$

$$= 1 \cdot (2 \ln x + 1) + x \cdot \frac{2}{x}$$

$$= 2 \ln x + 3$$

(3) $y = \frac{\cos x}{1 - \sin x}$

[Sol] $y' = \frac{(\cos x)'(1 - \sin x) - \cos x(1 - \sin x)'}{(1 - \sin x)^2}$

$$= \frac{-\sin x(1 - \sin x) + \cos^2 x}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$

$$\therefore y'' = -\frac{(1 - \sin x)'}{(1 - \sin x)^2}$$

$$= \frac{\cos x}{(1 - \sin x)^2}$$

#2

2. Find the n^{th} order derivatives of the following functions.

Ex.

$$y = e^{3x}$$

$$[\text{Sol}] \quad y' = e^{3x} \cdot (3x)' = 3e^{3x}$$

$$y'' = 3e^{3x} \cdot (3x)' = 3^2 e^{3x}$$

$$y''' = 3^2 e^{3x} \cdot (3x)' = 3^3 e^{3x}$$

$$\vdots$$

$$\therefore y^{(n)} = 3^n e^{3x}$$

(1) $y = e^{-2x}$

$$[\text{Sol}] \quad y' = e^{-2x} \cdot (-2x)' = -2e^{-2x}$$

$$y'' = -2e^{-2x} \cdot (-2x)' = (-2)^2 e^{-2x}$$

$$y''' = (-2)^2 e^{-2x} \cdot (-2x)' = (-2)^3 e^{-2x}$$

$$\vdots$$

$$\therefore y^{(n)} = (-2)^n e^{-2x}$$

(2) $y = xe^x$

$$[\text{Sol}] \quad y' = (x)'e^x + x(e^x)' = 1 \cdot e^x + xe^x = (x+1)e^x$$

$$y'' = (x+1)'e^x + (x+1)(e^x)' = 1 \cdot e^x + (x+1)e^x = (x+2)e^x$$

$$y''' = (x+2)'e^x + (x+2)(e^x)' = 1 \cdot e^x + (x+2)e^x = (x+3)e^x$$

$$\vdots$$

$$\therefore y^{(n)} = (x+n)e^x$$

(3) $y = x^a$

$$[\text{Sol}] \quad y' = ax^{a-1}$$

$$y'' = a(a-1)x^{a-2}$$

$$y''' = a(a-1)(a-2)x^{a-3}$$

$$\vdots$$

$$\therefore y^{(n)} = a(a-1)(a-2) \cdots (a-n+1)x^{a-n}$$

Differentiation of Various Functions
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Ex.Prove that function $y = e^{-x} \cos x$ satisfies the following equation.

$$y'' + 2y' + 2y = 0$$

$$[\text{Sol}] \quad y' = (e^{-x})' \cos x + e^{-x} (\cos x)'$$

$$= -e^{-x} \cos x - e^{-x} \sin x$$

$$= -e^{-x} (\cos x + \sin x)$$

$$y'' = -[(e^{-x})' (\cos x + \sin x) + e^{-x} (\cos x + \sin x)']$$

$$= e^{-x} (\cos x + \sin x) - e^{-x} (-\sin x + \cos x)$$

$$= 2e^{-x} \sin x$$

Therefore,

$$y'' + 2y' + 2y = 2e^{-x} \sin x - 2e^{-x} (\cos x + \sin x) + 2e^{-x} \cos x$$

$$= 2e^{-x} (\sin x - \cos x - \sin x + \cos x)$$

$$= 0$$

1. Prove that function $y = e^x \sin x$ satisfies the following equation.

$$y'' - 2y' + 2y = 0$$

$$[\text{Sol}] \quad y' = (e^x)' \sin x + e^x (\sin x)'$$

$$= e^x \sin x + e^x \cos x$$

$$= e^x (\sin x + \cos x)$$

$$y'' = (e^x)' (\sin x + \cos x) + e^x (\sin x + \cos x)'$$

$$= e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$$

$$= 2e^x \cos x$$

Therefore,

$$y'' - 2y' + 2y = 2e^x \cos x - 2e^x (\sin x + \cos x) + 2e^x \sin x$$

$$= 2e^x (\cos x - \sin x - \cos x + \sin x)$$

$$= 0$$

2. Prove that function $y = x + \sqrt{1+x^2}$ satisfies the following equation.

$$(1+x^2)y'' + xy' - y = 0$$

[Sol] $y' = 1 + \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x = 1 + \frac{x}{\sqrt{1+x^2}}$

$$\begin{aligned} y'' &= \frac{(x)' \sqrt{1+x^2} - x(\sqrt{1+x^2})'}{1+x^2} = \frac{1 \cdot \sqrt{1+x^2} - x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x}{1+x^2} \\ &= \frac{1}{(1+x^2)\sqrt{1+x^2}} \end{aligned}$$

Therefore,

$$\begin{aligned} (1+x^2)y'' + xy' - y &= \frac{1+x^2}{(1+x^2)\sqrt{1+x^2}} + x \left(1 + \frac{x}{\sqrt{1+x^2}} \right) - (x + \sqrt{1+x^2}) \\ &= \frac{1+x\sqrt{1+x^2} + x^2 - x\sqrt{1+x^2} - (1+x^2)}{\sqrt{1+x^2}} \\ &= 0 \end{aligned}$$

3. Find the values of constants a and b such that $f''(x) = af(x) + bf'(x)$ when $f(x) = e^{3x} \cos x$.

[Sol] $f'(x) = (e^{3x})' \cos x + e^{3x}(\cos x)'$

$$= 3e^{3x} \cos x - e^{3x} \sin x$$

$$= e^{3x}(3\cos x - \sin x)$$

$$f''(x) = (e^{3x})'(3\cos x - \sin x) + e^{3x}(3\cos x - \sin x)'$$

$$= 3e^{3x}(3\cos x - \sin x) + e^{3x}(-3\sin x - \cos x)$$

$$= e^{3x}(8\cos x - 6\sin x) \dots \textcircled{1}$$

$$af(x) + bf'(x) = ae^{3x} \cos x + be^{3x}(3\cos x - \sin x)$$

$$= e^{3x}[(a+3b)\cos x - b\sin x] \dots \textcircled{2}$$

From $\textcircled{1}$, $\textcircled{2}$ and $e^{3x} \neq 0$,

$$a+3b=8, b=6$$

$$\therefore a=-10, b=6$$

← Coefficient Comparison Method (J181)

Differentiation of Various Functions and Higher Order Derivatives

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Ex.

Using mathematical induction, prove that the n^{th} order derivative of function $y = \sin x$ is represented as follows:

$$y^{(n)} = \sin\left(x + \frac{n\pi}{2}\right) \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = y^{(1)} = (\sin x)' = \cos x$$

$$\leftarrow y^{(1)} = y'$$

$$\text{RHS} = \sin\left(x + \frac{1 \cdot \pi}{2}\right) = \cos x$$

$$\leftarrow \sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$y^{(k)} = \sin\left(x + \frac{k\pi}{2}\right) \dots \textcircled{2}$$

When $n=k+1$, differentiating both sides of $\textcircled{2}$ with respect to x ,

$$y^{(k+1)} = \cos\left(x + \frac{k\pi}{2}\right)$$

$$= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$$

$$\cos \theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

$$= \sin\left[x + \frac{(k+1)\pi}{2}\right]$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

ANSWERS: $\cos x$, $\cos x$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$

N188b

1. Using mathematical induction, prove that the n^{th} order derivative of function $y = \ln x$ is represented as follows:

$$y^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n} \dots \textcircled{1}$$

$$((n-1)! = (n-1)(n-2) \dots 3 \cdot 2 \cdot 1, 0! = 1)$$

[Sol] (i) When $n=1$,

$$\text{LHS} = y^{(1)} = (\ln x)' = \frac{1}{x}$$

$$\text{RHS} = (-1)^0 \cdot \frac{0!}{x^1} = \frac{1}{x} \quad \leftarrow a^0 = 1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$y^{(k)} = (-1)^{k-1} \cdot \frac{(k-1)!}{x^k} \dots \textcircled{2}$$

When $n=k+1$, differentiating both sides of $\textcircled{2}$ with respect to x ,

$$y^{(k+1)} = (-1)^{k-1} \cdot \left[-\frac{(k-1)! \cdot kx^{k-1}}{x^{2k}} \right]$$

$$= (-1)^{k-1} \cdot (-1) \cdot \frac{(k-1)! \cdot k}{x^{2k-(k-1)}}$$

$$= (-1)^k \cdot \frac{k!}{x^{k+1}}$$

$$= (-1)^{(k+1)-1} \cdot \frac{[(k+1)-1]!}{x^{k+1}}$$

$$\begin{aligned} & (k-1)! \cdot k \\ &= (k-1)(k-2) \dots 3 \cdot 2 \cdot 1 \cdot k \\ &= k! \end{aligned}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Differentiation of Various Functions
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1. Given that y is a function of x , express $\frac{d^2y}{dx^2}$ in terms of x and y for the following equation.

$$x^2 - y^2 = a^2 \quad (a \text{ is a constant})$$

[Sol] Differentiating both sides with respect to x ,

$$2x - 2y \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{x}{y}$$

Furthermore, differentiating both sides with respect to x ,

$$\frac{d^2y}{dx^2} = \frac{(x)'y - x(y)'}{y^2} = \frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2} = \frac{y^2 - x^2}{y^3}$$

Since $\frac{dy}{dx} = \frac{x}{y}$

2. Using the equality $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$, express $\frac{d^2y}{dx^2}$ in terms of t when $x = 3t^3$ and $y = 9t + 1$.

[Sol] $\frac{dx}{dt} = 9t^2$, $\frac{dy}{dt} = 9$

$$\therefore \frac{dy}{dx} = \frac{9}{9t^2} = \frac{1}{t^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{1}{t^2} \right) \cdot \frac{dt}{dx} = -\frac{2}{t^3} \cdot \frac{1}{9t^2} = -\frac{2}{9t^5}$$

$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{9t^2}$

The second, third and n^{th} order derivatives are often expressed as

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^ny}{dx^n} \text{ or } \frac{d^2}{dx^2}f(x), \frac{d^3}{dx^3}f(x), \frac{d^n}{dx^n}f(x) \text{ respectively.}$$

$\frac{d^2y}{dx^2}$ is read as "d two y by dx squared."

3. Given $f(x) = \cos x + 1$ and $g(x) = \frac{a}{bx^2 + cx + 1}$, find the values of constants a , b and c which satisfy $f(0) = g(0)$, $f'(0) = g'(0)$ and $f''(0) = g''(0)$.

[Sol] $f(0) = \cos 0 + 1 = 2$, $g(0) = a$

Since $f(0) = g(0)$, $a = 2 \dots \textcircled{1}$

From $\textcircled{1}$, $g(x) = \frac{2}{bx^2 + cx + 1}$

Since $g'(x) = -\frac{2(2bx + c)}{(bx^2 + cx + 1)^2}$, $g'(0) = -2c$

Also, since $f'(x) = -\sin x$, $f'(0) = 0$

Since $f'(0) = g'(0)$, $c = 0 \dots \textcircled{2}$

From $\textcircled{2}$, $g'(x) = -\frac{4bx}{(bx^2 + 1)^2}$

$$g''(x) = -\frac{4b(bx^2 + 1)^2 - 4bx \cdot 2(bx^2 + 1) \cdot 2bx}{(bx^2 + 1)^4}$$

$$= -\frac{4b(bx^2 + 1 - 4bx^2)}{(bx^2 + 1)^3}$$

$$= \frac{4b(3bx^2 - 1)}{(bx^2 + 1)^3}$$

$\therefore g''(0) = -4b$

Also, since $f''(x) = -\cos x$, $f''(0) = -1$

Since $f''(0) = g''(0)$, $b = \frac{1}{4}$

$\therefore a = 2$, $b = \frac{1}{4}$, $c = 0$

Differentiation of Various Functions and Higher Order Derivatives

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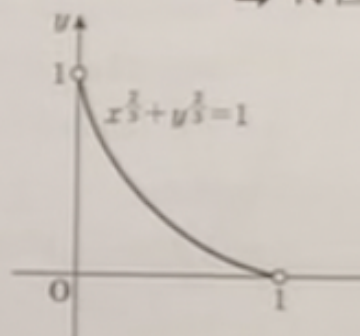
1. Given $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, express $\frac{dy}{dx}$ in terms of x and y . ($x > 0$, $y > 0$)

➡ N13

[Sol] Differentiating both sides with respect to x ,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}} \quad \left[= -\left(\frac{x}{y}\right)^{-\frac{1}{3}} \right]$$



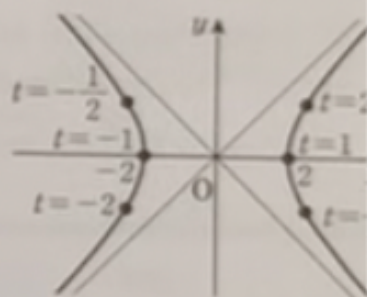
2. Given $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$, express $\frac{dy}{dx}$ in terms of t .

➡ N18

[Sol] $\frac{dx}{dt} = 1 - \frac{1}{t^2}$, $\frac{dy}{dt} = 1 + \frac{1}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

$$\left[= \frac{t^2 + 1}{(t+1)(t-1)} \right]$$



3. Find the second and third order derivatives of function $y = x^3 \ln x$. ➡ N18

[Sol] $y' = (x^3)' \ln x + x^3 (\ln x)'$

$$= 3x^2 \ln x + x^3 \cdot \frac{1}{x}$$

$$= x^2 (3 \ln x + 1)$$

$$\therefore y'' = (x^2)' (3 \ln x + 1) + x^2 (3 \ln x + 1)'$$

$$= 2x (3 \ln x + 1) + x^2 \cdot \frac{3}{x}$$

$$= 6x \ln x + 5x \quad \left[= x(6 \ln x + 5) \right]$$

$$y''' = 6[(x)' \ln x + x (\ln x)'] + 5$$

$$= 6\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) + 5$$

$$= 6 \ln x + 11$$

4. Prove that function $y = a \cos kx + b \sin kx$ satisfies $y'' + k^2 y = 0$. $\Rightarrow$ N187

$$\begin{aligned}
 [\text{Sol}] \quad y' &= -a \sin kx \cdot (kx)' + b \cos kx \cdot (kx)' \\
 &= -ak \sin kx + bk \cos kx \\
 &= -k(a \sin kx - b \cos kx) \\
 y'' &= -k[a \cos kx \cdot (kx)' + b \sin kx \cdot (kx)'] \\
 &= -k(ak \cos kx + bk \sin kx) \\
 &= -k^2(a \cos kx + b \sin kx)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 y'' + k^2 y &= -k^2(a \cos kx + b \sin kx) + k^2(a \cos kx + b \sin kx) \\
 &= 0
 \end{aligned}$$

5. Using mathematical induction, prove that the n^{th} order derivative of function $y = xe^{-x}$ is represented as follows: $\Rightarrow$ N188

$$y^{(n)} = (-1)^n (x - n) e^{-x} \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = y^{(1)} = (x)' e^{-x} + x(e^{-x})' = 1 \cdot e^{-x} - x e^{-x} = -(x-1) e^{-x}$$

$$\text{RHS} = (-1)^1 \cdot (x-1) e^{-x} = -(x-1) e^{-x}$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$y^{(k)} = (-1)^k (x-k) e^{-x} \dots \textcircled{2}$$

When $n=k+1$, differentiating both sides of $\textcircled{2}$ with respect to x ,

$$y^{(k+1)} = (-1)^k [(x-k)' e^{-x} + (x-k)(e^{-x})']$$

$$= (-1)^k [1 \cdot e^{-x} - (x-k) e^{-x}]$$

$$= (-1)^k \cdot (-1) \cdot (x-k-1) e^{-x}$$

$$= (-1)^{k+1} [x-(k+1)] e^{-x}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Various Properties of Derivatives

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Examine if the following functions are differentiable at the points indicated in the brackets [].

Ex.

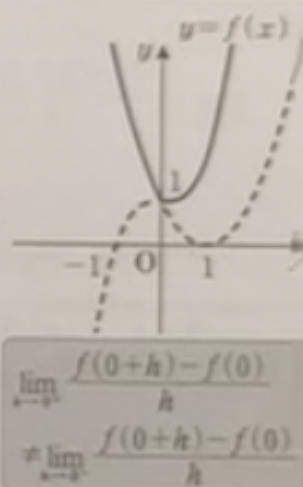
$$f(x) = \begin{cases} x^3 + 1 & (x \geq 0) \\ (x-1)^2 & (x < 0) \end{cases} \quad [x=0]$$

$$\begin{aligned} [\text{Sol}] \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{[(0+h)^3 + 1] - 1}{h} \\ &= \lim_{h \rightarrow 0^+} h^2 = 0 \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{[(0+h)-1]^2 - 1}{h} \\ &= \lim_{h \rightarrow 0^-} (h-2) = -2 \end{aligned}$$

Therefore, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Thus, $f(x)$ is not differentiable at $x=0$.



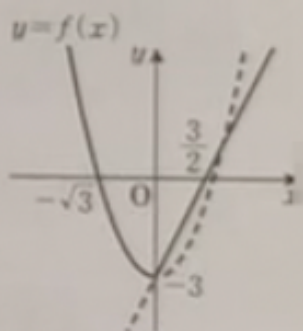
$$(1) f(x) = \begin{cases} 2x-3 & (x \geq 0) \\ x^2-3 & (x < 0) \end{cases} \quad [x=0]$$

$$\begin{aligned} [\text{Sol}] \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{[2(0+h)-3] - (-3)}{h} \\ &= \lim_{h \rightarrow 0^+} 2 = 2 \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{[(0+h)^2 - 3] - (-3)}{h} \\ &= \lim_{h \rightarrow 0^-} h = 0 \end{aligned}$$

Therefore, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Thus, $f(x)$ is not differentiable at $x=0$.



[Reference] N141

Given the function $f(x)$, if the limit value $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists, then $f(x)$ is differentiable at $x=a$.

$$(2) \quad f(x) = |x-2| \quad [x=2]$$

[Sol] When $x \geq 2$, $f(x) = x-2$

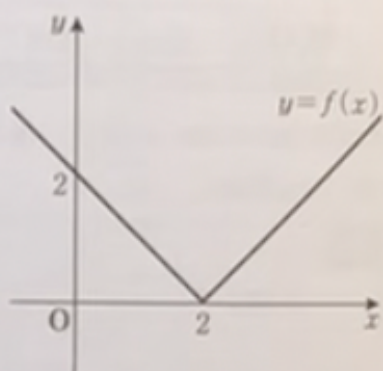
When $x < 2$, $f(x) = -(x-2)$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{[(2+h)-2] - 0}{h} \\ &= \lim_{h \rightarrow 0^+} 1 = 1 \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^-} \frac{-[(2+h)-2] - 0}{h} \\ &= \lim_{h \rightarrow 0^-} (-1) = -1 \end{aligned}$$

Therefore, $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$ does not exist.

Thus, $f(x)$ is **not differentiable** at $x=2$.



$$(3) \quad f(x) = x|x| \quad [x=0]$$

[Sol] When $x \geq 0$, $f(x) = x^2$

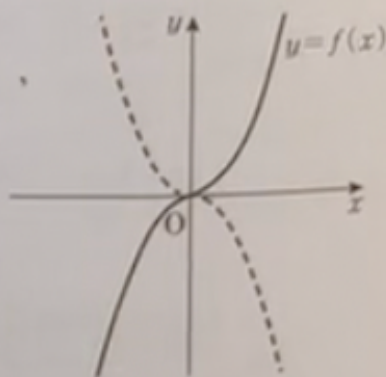
When $x < 0$, $f(x) = -x^2$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{(0+h)^2 - 0}{h} \\ &= \lim_{h \rightarrow 0^+} h = 0 \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{-(0+h)^2 - 0}{h} \\ &= \lim_{h \rightarrow 0^-} (-h) = 0 \end{aligned}$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 0 \quad \leftarrow \quad \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h}$$

Therefore, $f(x)$ is **differentiable** at $x=0$.



Various Properties of Derivatives

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If the function $f(x)$ is differentiable at $x=a$, then $f'(a)$ exists and

$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] = f'(a) \cdot 0 = 0$$

$$\therefore \lim_{x \rightarrow a} f(x) = f(a) \quad \leftarrow f(x) \text{ is continuous at } x=a.$$

Therefore, the following statement is true.

Differentiability and Continuity

If the function $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$.

Examine if the following functions are continuous and also differentiable at the points indicated in the brackets ().

Ex. $f(x) = |x|$ ($x=0$)

[Sol] When $x \geq 0$, $f(x) = x$

When $x < 0$, $f(x) = -x$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

Examining if $\lim_{x \rightarrow a} f(x)$ exists

$$\text{Also, since } f(0) = 0, \lim_{x \rightarrow 0} f(x) = f(0)$$

Examining if $\lim_{x \rightarrow a} f(x) = f(a)$ is true

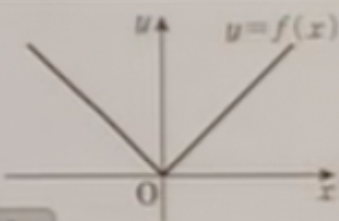
Therefore, $f(x)$ is continuous at $x=0$.

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(0+h) - 0}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-(0+h) - 0}{h} = \lim_{h \rightarrow 0^-} (-1) = -1$$

Thus, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Therefore, $f(x)$ is not differentiable at $x=0$.



From **Ex.**, even if the function $f(x)$ is continuous at $x=a$, it is not always differentiable at $x=a$.

$$(1) f(x) = |x(x-2)| \quad [x=2]$$

[Sol] When $x(x-2) \geq 0$, i.e. $x \leq 0$, $2 \leq x$, $f(x) = x(x-2)$

When $x(x-2) < 0$, i.e. $0 < x < 2$, $f(x) = -x(x-2)$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x(x-2) = 0$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} [-x(x-2)] = 0$$

Also, since $f(2) = 0$, $\lim_{x \rightarrow 2} f(x) = f(2)$

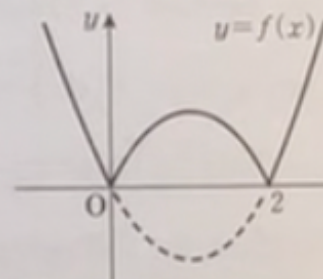
Therefore, $f(x)$ is **continuous** at $x=2$.

$$\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0^+} \frac{(2+h)[(2+h)-2] - 0}{h} = \lim_{h \rightarrow 0^+} (2+h) = 2$$

$$\lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0^-} \frac{-(2+h)[(2+h)-2] - 0}{h} = \lim_{h \rightarrow 0^-} [-(2+h)] = -$$

Thus, $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$ does not exist.

Therefore, $f(x)$ is **not differentiable** at $x=2$.



$$(2) f(x) = \begin{cases} x^2 & (x \geq 1) \\ 2x-1 & (x < 1) \end{cases} \quad [x=1]$$

[Sol] $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x^2 = 1$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (2x-1) = 1$$

Also, since $f(1) = 1$, $\lim_{x \rightarrow 1} f(x) = f(1)$

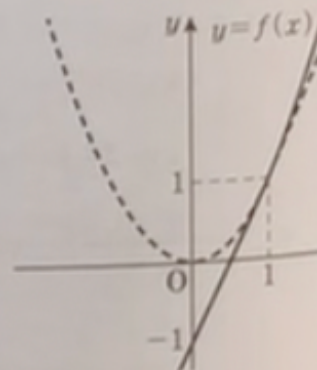
Therefore, $f(x)$ is **continuous** at $x=1$.

$$\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \rightarrow 0^+} (2+h) = 2$$

$$\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{[2(1+h)-1] - 1}{h} = \lim_{h \rightarrow 0^-} 2 = 2$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = 2$$

Thus, $f(x)$ is **differentiable** at $x=1$.



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Examine if the following functions are continuous and also differentiable at the points indicated in the brackets [].

$$(1) f(x) = \begin{cases} x^3 + 7 & (x \geq 0) \\ x + 7 & (x < 0) \end{cases} \quad [x=0]$$

$$[\text{Sol}] \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x^3 + 7) = 7$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 7) = 7$$

$$\text{Also, since } f(0) = 7, \lim_{x \rightarrow 0} f(x) = f(0)$$

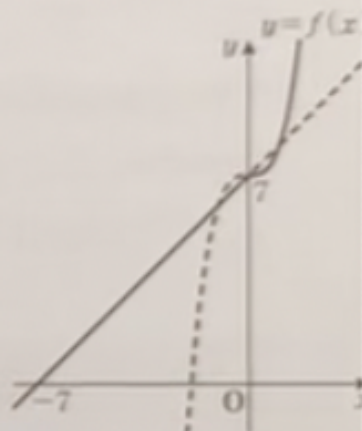
Therefore, $f(x)$ is **continuous** at $x=0$.

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{[(0+h)^3 + 7] - 7}{h} = \lim_{h \rightarrow 0^+} h^2 = 0$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{[(0+h) + 7] - 7}{h} = \lim_{h \rightarrow 0^-} 1 = 1$$

$$\text{Thus, } \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \text{ does not exist.}$$

Therefore, $f(x)$ is **not differentiable** at $x=0$.



$$(2) f(x) = \sqrt[3]{x^2} \quad [x=0]$$

$$[\text{Sol}] \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sqrt[3]{x^2} = 0$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sqrt[3]{x^2} = 0$$

$$\text{Also, since } f(0) = 0, \lim_{x \rightarrow 0} f(x) = f(0)$$

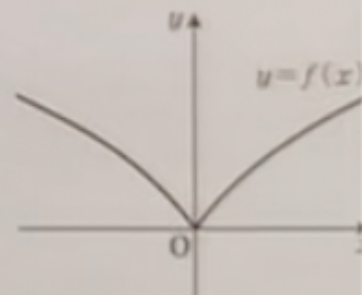
Therefore, $f(x)$ is **continuous** at $x=0$.

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt[3]{(0+h)^2} - 0}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt[3]{h}} = \infty$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{\sqrt[3]{(0+h)^2} - 0}{h} = \lim_{h \rightarrow 0^-} \frac{1}{\sqrt[3]{h}} = -\infty$$

$$\text{Thus, } \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \text{ does not exist.}$$

Therefore, $f(x)$ is **not differentiable** at $x=0$.



$$(3) \quad f(x) = \begin{cases} \sin x & (x \geq 0) \\ \frac{1}{2}x^2 + x & (x < 0) \end{cases} \quad [x=0]$$

[Sol] $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sin x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(\frac{1}{2}x^2 + x \right) = 0$$

Also, since $f(0) = 0$, $\lim_{x \rightarrow 0} f(x) = f(0)$

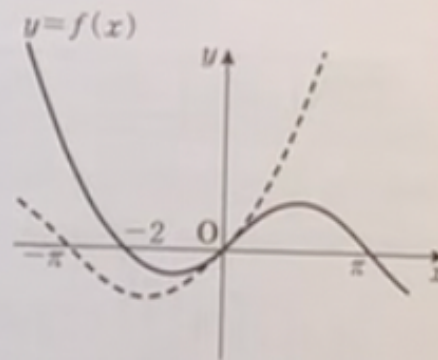
Therefore, $f(x)$ is **continuous** at $x=0$.

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{\sin(0+h) - 0}{h} = \lim_{h \rightarrow 0^+} \frac{\sin h}{h} = 1 \quad \leftarrow$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{\left[\frac{1}{2}(0+h)^2 + (0+h) \right] - 0}{h} = \lim_{h \rightarrow 0^-} \left(\frac{h}{2} + 1 \right) = 1$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 1$$

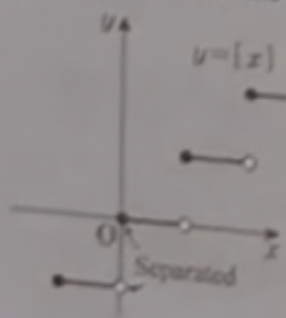
Thus, $f(x)$ is **differentiable** at $x=0$.



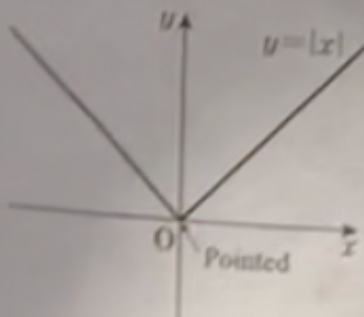
$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Note Summary

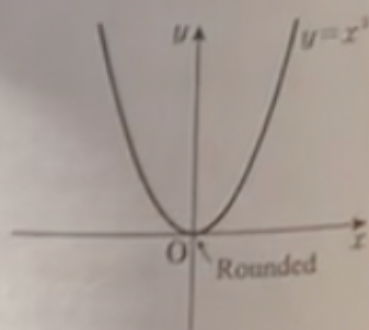
Relationship between continuity and differentiability is as shown below.
(Pay attention to the origin.)



Not continuous and
not differentiable
(Not connected)



Continuous but
not differentiable
(Connected but not smooth)



Continuous and
differentiable
(Connected and smooth)

Various Properties of Derivatives

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Ex.

Ex. The function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^3 + ax & (x \geq 2) \\ bx^2 - ax & (x < 2) \end{cases}$$

Find the values of a and b such that $f(x)$ is differentiable at $x=2$.

[Sol] If the function $f(x)$ is differentiable at $x=2$,

$f(x)$ is continuous at $x=2$; therefore, $\lim_{x \rightarrow 2} f(x) = f(2)$

$$\therefore \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\therefore b \cdot 2^2 - a \cdot 2 = 2^3 + a \cdot 2 \leftarrow$$

$$\therefore b-a=2 \cdots \textcircled{1}$$

Since $f(x) = x^3 + ax$ when $x \geq 2$,
 $\lim_{x \rightarrow 2^+} f(x) = f(2)$ is obvious. Therefore,
 considering $\lim_{x \rightarrow 2} f(x) = f(2)$

Also,

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{[(2+h)^3 + a(2+h)] - (8+2a)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h^3 + 6h^2 + (12+a)h}{h} \\ &= \lim_{h \rightarrow 0^+} (h^2 + 6h + 12+a) \\ &= 12+a\end{aligned}$$

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0} \frac{[b(2+h)^2 - a(2+h)] - (8+2a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{bh^2 + (4b-a)h + 4(b-a-2)}{h} \\ &= \lim_{h \rightarrow 0} (bh + 4b - a) \\ &= 4b - a\end{aligned}$$

From ①

$f(x)$ is differentiable at $x=2$ when $12+a = 4b-a \dots \textcircled{2}$

From ① and ②,

$$a = \boxed{2}, b = \boxed{4}$$

1. The function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{ax+b}{x+1} & (x > 1) \\ x^2+1 & (x \leq 1) \end{cases}$$

Find the values of a and b such that $f(x)$ is differentiable at $x=1$.

[Sol] If the function $f(x)$ is differentiable at $x=1$,

$f(x)$ is continuous at $x=1$; therefore, $\lim_{x \rightarrow 1} f(x) = f(1)$

$$\therefore \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

$$\therefore \frac{a \cdot 1 + b}{1+1} = 1^2 + 1 \quad \leftarrow$$

$$\therefore a + b = 4 \quad \dots \textcircled{1}$$

Since $f(x) = x^2 + 1$ when $x \leq 1$,
 $\lim_{x \rightarrow 1} f(x) = f(1)$ is obvious. Therefore,
 considering $\lim_{x \rightarrow 1} f(x) = f(1)$

Also,

$$\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{a(1+h)+b}{(1+h)+1} - 2}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{a+b-4+(a-2)h}{h(2+h)}$$

$$= \lim_{h \rightarrow 0^+} \frac{a-2}{2+h}$$

$$= \frac{a-2}{2}$$

From ①

$$\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{[(1+h)^2 + 1] - 2}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{2h + h^2}{h}$$

$$= \lim_{h \rightarrow 0^-} (2+h)$$

$$= 2$$

$f(x)$ is differentiable at $x=1$ when $\frac{a-2}{2} = 2 \quad \dots \textcircled{2}$

From ① and ②,

$$a = 6, \quad b = -2$$

Various Properties of Derivatives

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1. Given that the function $f(x)$ is differentiable at $x=a$, express the following limit values in terms of $f'(a)$.

Ex.

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{f(a+3h) - f(a+h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(a+3h) - f(a) + f(a) - f(a+h)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{f(a+3h) - f(a)}{3h} \cdot 3 - \frac{f(a+h) - f(a)}{h} \right] \\
 &= 3f'(a) - f'(a) \\
 &= 2f'(a)
 \end{aligned}$$

Rearranging into the form

$$\lim_{h \rightarrow 0} \frac{f(a + \bullet h) - f(a)}{\bullet h}$$

$$\begin{aligned}
 (1) \quad & \lim_{h \rightarrow 0} \frac{f(a+3h) - f(a+2h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(a+3h) - f(a) + f(a) - f(a+2h)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{f(a+3h) - f(a)}{3h} \cdot 3 - \frac{f(a+2h) - f(a)}{2h} \cdot 2 \right] \\
 &= 3f'(a) - 2f'(a) \\
 &= f'(a)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a-h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a) + f(a) - f(a-h)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{f(a+2h) - f(a)}{2h} \cdot 2 - \frac{f(a-h) - f(a)}{-h} \cdot (-1) \right] \\
 &= 2f'(a) + f'(a) \\
 &= 3f'(a)
 \end{aligned}$$

2. Given that the function $f(x)$ is differentiable at $x=a$, express the following limit values in terms of a , $f(a)$ and $f'(a)$.

Ex.

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{af(x) - xf(a)}{x-a} \\ &= \lim_{x \rightarrow a} \frac{af(x) - af(a) + af(a) - xf(a)}{x-a} \\ &= \lim_{x \rightarrow a} \left\{ \frac{a[f(x) - f(a)]}{x-a} - \frac{(x-a)f(a)}{x-a} \right\} \quad \leftarrow \text{Rearranging into the form } \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a} \\ &= af'(a) - f(a) \end{aligned}$$

$$\begin{aligned} (1) \quad & \lim_{x \rightarrow a} \frac{x^2 f(x) - a^2 f(a)}{x-a} \\ &= \lim_{x \rightarrow a} \frac{x^2 f(x) - x^2 f(a) + x^2 f(a) - a^2 f(a)}{x-a} \\ &= \lim_{x \rightarrow a} \left\{ \frac{x^2 [f(x) - f(a)]}{x-a} + \frac{(x+a)(x-a)f(a)}{x-a} \right\} \\ &= \lim_{x \rightarrow a} \left\{ \frac{x^2 [f(x) - f(a)]}{x-a} + (x+a)f(a) \right\} \\ &= a^2 f'(a) + 2af(a) \quad [=a[af'(a) + 2f(a)]] \end{aligned}$$

$$\begin{aligned} (2) \quad & \lim_{x \rightarrow a} \frac{[af(x)]^2 - [xf(a)]^2}{x-a} \\ &= \lim_{x \rightarrow a} \frac{[af(x)]^2 - [af(a)]^2 + [af(a)]^2 - [xf(a)]^2}{x-a} \\ &= \lim_{x \rightarrow a} \left\{ \frac{a^2 [f(x) + f(a)][f(x) - f(a)]}{x-a} - \frac{(x+a)(x-a)[f(a)]^2}{x-a} \right\} \\ &= \lim_{x \rightarrow a} \left\{ \frac{a^2 [f(x) + f(a)][f(x) - f(a)]}{x-a} - (x+a)[f(a)]^2 \right\} \\ &= 2a^2 f(a) f'(a) - 2a[f(a)]^2 \quad [=2af(a)[af'(a) - f(a)]] \end{aligned}$$

Alternative Solution

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{[af(x)]^2 - [xf(a)]^2}{x-a} = \lim_{x \rightarrow a} \frac{[af(x) + xf(a)][af(x) - xf(a)]}{x-a} \\ & \lim_{x \rightarrow a} [af(x) + xf(a)] = 2af(a), \quad \lim_{x \rightarrow a} \frac{af(x) - xf(a)}{x-a} = af'(a) - f(a) \quad \leftarrow \text{From Ex.} \\ & \therefore \lim_{x \rightarrow a} \frac{[af(x)]^2 - [xf(a)]^2}{x-a} = 2af(a)[af'(a) - f(a)] \\ & \quad \quad \quad [=2a^2 f(a) f'(a) - 2a[f(a)]^2] \end{aligned}$$

Various Properties of Derivatives

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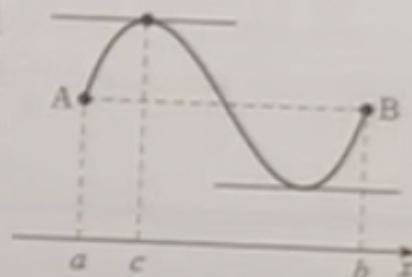
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Rolle's Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one value c such that $f'(c) = 0$ and $a < c < b$.



This theorem states that, if $f(a) = f(b)$, then there exists at least one point between A and B on the curve at which the slope of the tangent is 0, i.e. $f'(c) = 0$, as shown in the diagram above.

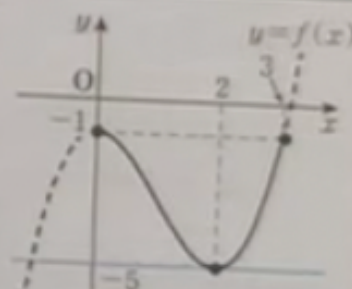
Find the value of c that satisfies Rolle's Theorem for each given function and closed interval.

Ex. $f(x) = x^3 - 3x^2 - 1$ $[0, 3]$

[Sol] $f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f'(c) = 3c(c - 2) = 0$

Since $0 < c < 3$, $c = 2$ ← $c = 0$ is not appropriate.

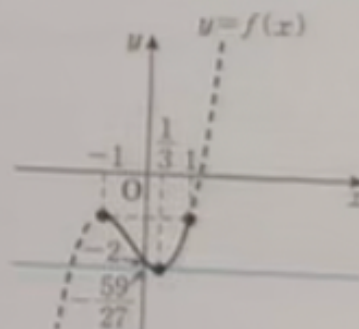


(1) $f(x) = x^3 + x^2 - x - 2$ $[-1, 1]$

[Sol] $f'(x) = 3x^2 + 2x - 1 = (x + 1)(3x - 1)$

$f'(c) = (c + 1)(3c - 1) = 0$

Since $-1 < c < 1$, $c = \frac{1}{3}$

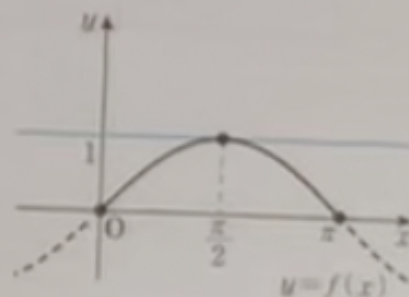


(2) $f(x) = \sin x$ $[0, \pi]$

[Sol] $f'(x) = \cos x$

$f'(c) = \cos c = 0$

Since $0 < c < \pi$, $c = \frac{\pi}{2}$



N196b

Using Rolle's Theorem, prove the following statement.

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists at least one value c such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ and } a < c < b.$$

[Sol] Let $\frac{f(b)-f(a)}{b-a} = k$. Then show that $f'(c) = k$.

Rearranging $\frac{f(b)-f(a)}{b-a} = k$, $f(b) - f(a) - k(b-a) = 0 \dots \textcircled{1}$

Consider the function

$$F(x) = f(x) - f(a) - k(x-a) \dots \textcircled{2}$$

which is derived by changing b of the LHS of $\textcircled{1}$ to x .

From $\textcircled{2}$, $F(a) = \boxed{0}$

From $\textcircled{1}$, $F(b) = \boxed{0}$

$$\therefore F(a) = F(b)$$

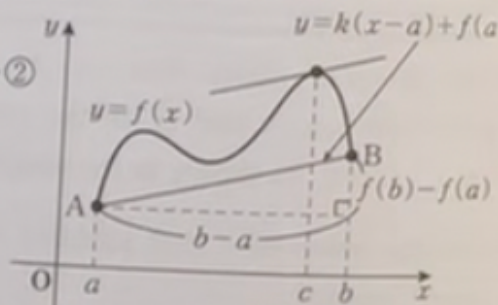
Also, differentiating both sides of $\textcircled{2}$ with respect to x ,

$$F'(x) = f'(x) - \boxed{k} \dots \textcircled{3}$$

Therefore, since the function $F(x)$ satisfies the conditions of Rolle's Theorem, there exists at least one value c such that $F'(c) = 0$ and $a < c < b$.

From $\textcircled{3}$, $F'(c) = f'(c) - \boxed{k}$; therefore, $f'(c) = \boxed{k}$.

Thus, there exists at least one value c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$ and $a < c < b$.



Answers: 0, 0, k, k, k

Various Properties of Derivatives

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Mean Value Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists at least one value c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$ and $a < c < b$.

Find the value of c that satisfies the Mean Value Theorem for each given function and closed interval.

Ex. $f(x) = x^3 - 3x^2 + 2x$ $[0, 3]$

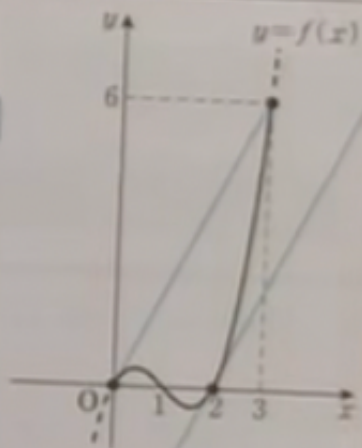
[Sol] $\frac{f(3)-f(0)}{3-0} = \frac{6-0}{3} = 2 \leftarrow a=0, b=3$

$$f'(x) = 3x^2 - 6x + 2$$

From the Mean Value Theorem, there exists at least one value c such that $2 = 3c^2 - 6c + 2$ and $0 < c < 3$.

$$3c(c-2) = 0$$

Since $0 < c < 3$, $c = 2 \leftarrow c=0$ is not appropriate.



(1) $f(x) = \frac{1}{x}$ $[1, 2]$

[Sol] $\frac{f(2)-f(1)}{2-1} = \frac{1}{2} - 1 = -\frac{1}{2}$

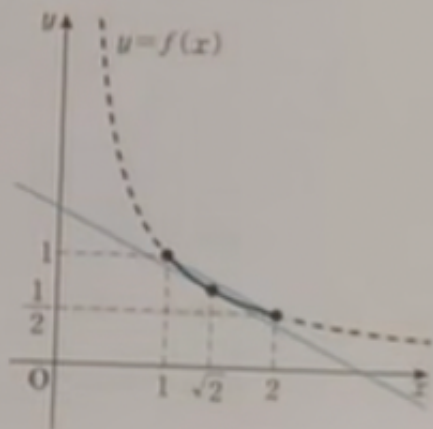
$$f'(x) = -\frac{1}{x^2}$$

From the Mean Value Theorem, there exists at least one value c such that

$$-\frac{1}{2} = -\frac{1}{c^2} \text{ and } 1 < c < 2.$$

$$c^2 = 2$$

Since $1 < c < 2$, $c = \sqrt{2}$



N197b

(2) $f(x) = \ln x$ $[1, 2]$ ($\ln 2 \approx 0.69$)

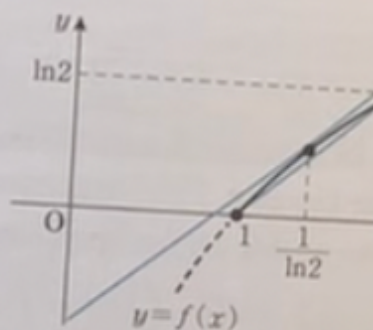
[Sol] $\frac{f(2) - f(1)}{2 - 1} = \ln 2 - \ln 1 = \ln 2$

$$f'(x) = \frac{1}{x}$$

From the Mean Value Theorem, there exists at least one value c such that

$$\ln 2 = \frac{1}{c} \text{ and } 1 < c < 2. \quad \leftarrow \ln 2 \approx 0.69$$

$$\therefore c = \frac{1}{\ln 2} \quad \leftarrow 1 < c < 2 \text{ is satisfied.}$$



(3) $f(x) = e^{-x}$ $[0, 1]$

[Sol] $\frac{f(1) - f(0)}{1 - 0} = \frac{1}{e} - 1$

$$f'(x) = -e^{-x}$$

From the Mean Value Theorem, there exists at least one value c such that

$$\frac{1}{e} - 1 = -e^{-c} \text{ and } 0 < c < 1.$$

$$e^{-c} = \frac{e-1}{e}$$

$$-c = \ln \frac{e-1}{e}$$

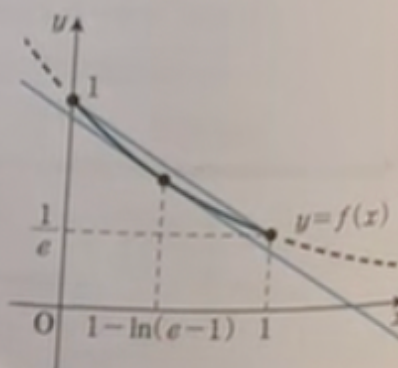
$$= \ln(e-1) - 1$$

$$\therefore c = 1 - \ln(e-1)$$

When $a^x = x$, $y = \log_a x$

$$\ln \frac{M}{N} = \ln M - \ln N, \quad \ln e = 1$$

Since $0 < \ln(e-1) < 1$,
 $0 < c < 1$ is satisfied.



Various Properties of Derivatives

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Using the Mean Value Theorem, prove the following inequalities.

Ex.

When $0 < a < b$, $\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$

[Sol] Let $f(x) = \ln x$.

$f(x)$ is differentiable at $x > 0$ and $f'(x) = \frac{1}{x}$.

Using the Mean Value Theorem on the interval $[a, b]$, there exists at least one value c such that $\frac{\ln b - \ln a}{b - a} = \frac{1}{c}$ and $a < c < b$. ←

Since $0 < a < c < b$, $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$ ←

$$\therefore \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

$$\frac{f(b) - f(a)}{b - a} = f'(c), a < c < b$$

$$\text{When } 0 < a < b < c, \frac{1}{a} > \frac{1}{b} > \frac{1}{c}$$

(1) When $a > 0$, $\frac{1}{a+1} < \ln(a+1) - \ln a < \frac{1}{a}$

[Sol] Let $f(x) = \ln x$.

$f(x)$ is differentiable at $x > 0$ and $f'(x) = \frac{1}{x}$.

Using the Mean Value Theorem on the interval $[a, a+1]$, there exists at least one value c such that $\frac{\ln(a+1) - \ln a}{(a+1) - a} = \frac{1}{c}$ and $a < c < a+1$.

Since $0 < a < c < a+1$, $\frac{1}{a+1} < \frac{1}{c} < \frac{1}{a}$

$$\therefore \frac{1}{a+1} < \ln(a+1) - \ln a < \frac{1}{a}$$

N198b

- (2) When $0 \leq a < b$ and $n \geq 2$, $b^n - a^n < nb^{n-1}(b-a)$

[Sol] Let $f(x) = x^n$.

$f(x)$ is differentiable for all real numbers x and $f'(x) = nx^{n-1}$.

Using the Mean Value Theorem on the interval $[a, b]$, there exists at least one value c such that $\frac{b^n - a^n}{b - a} = nc^{n-1}$ and $a < c < b$.

Since $0 \leq a < c < b$ and $n \geq 2$, $nc^{n-1} < nb^{n-1}$

$$\therefore \frac{b^n - a^n}{b - a} < nb^{n-1}$$

$$\therefore b^n - a^n < nb^{n-1}(b - a)$$

- (3) When $0 < \alpha < \beta < \frac{\pi}{2}$, $\sin \beta - \sin \alpha < \beta - \alpha$

[Sol] Let $f(x) = \sin x$.

$f(x)$ is differentiable for all real numbers x and $f'(x) = \cos x$.

Using the Mean Value Theorem on the interval $[\alpha, \beta]$, there exists at least one value c such that $\frac{\sin \beta - \sin \alpha}{\beta - \alpha} = \cos c$ and $\alpha < c < \beta$.

Since $0 < \alpha < c < \beta < \frac{\pi}{2}$, $\cos c < 1$

$$\therefore \frac{\sin \beta - \sin \alpha}{\beta - \alpha} < 1$$

$$\therefore \sin \beta - \sin \alpha < \beta - \alpha$$

When $0 < \theta < \frac{\pi}{2}$,

$$1 = \cos 0 > \cos \theta > \cos \frac{\pi}{2} = 0$$

Various Properties of Derivatives

Name _____

Date / /

Time : to :

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1. Given that $f(x)$ is a differentiable function that satisfies
 $f(-x) = f(x) + 2x$, $f'(1) = 1$ and $f(1) = 0$, solve the following questions.

- (1) Find the value of $f'(-1)$.

[Sol] Differentiating both sides of $f(-x) = f(x) + 2x$ with respect to x ,

$$-f'(-x) = f'(x) + 2 \quad \cdots \textcircled{1}$$

Substituting $x=1$ into $\textcircled{1}$,

$$\begin{aligned} -f'(-1) &= f'(1) + 2 \\ &= 3 \end{aligned}$$

$$\therefore f'(-1) = -3$$

$$\begin{aligned} [f(-x)]' &= f'(-x) \cdot (-x)' \\ &= -f'(-x) \end{aligned}$$

$$f'(1) = 1$$

- (2) Find the value of $\lim_{x \rightarrow 1} \frac{f(x) + f(-x) - 2}{x - 1}$.

$$[\text{Sol}] \lim_{x \rightarrow 1} \frac{f(x) + f(-x) - 2}{x - 1} = \lim_{x \rightarrow 1} \frac{f(x) + [f(x) + 2x] - 2}{x - 1}$$

$$\begin{aligned} f(-x) &= f(x) + 2x \end{aligned}$$

$$= \lim_{x \rightarrow 1} \frac{2f(x) + 2(x - 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} \left\{ \frac{2[f(x) - f(1)]}{x - 1} + 2 \right\}$$

$$f(1) = 0$$

$$= 2f'(1) + 2$$

$$= 4$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Alternative Solution

$$\text{Let } F(x) = f(x) + f(-x) - 2.$$

$$F(1) = f(1) + f(-1) - 2 = 0$$

$$F'(x) = f'(x) - f'(-x)$$

$$\therefore \lim_{x \rightarrow 1} \frac{f(x) + f(-x) - 2}{x - 1} = \lim_{x \rightarrow 1} \frac{F(x) - F(1)}{x - 1} = F'(1) = f'(1) - f'(-1) = 4$$

$$\begin{aligned} \text{Substituting } x=1 \text{ into } f(-x) &= f(x) + 2x, \\ f(-1) &= f(1) + 2 \cdot 1 = 2 \end{aligned}$$

2. Using the Mean Value Theorem, find $\lim_{x \rightarrow 0} \frac{\sin x - \sin x^2}{x - x^2}$.

[Sol] Let $f(x) = \sin x$.

$f(x)$ is differentiable for all real numbers x and $f'(x) = \cos x$.

(i) When $x < 0$, $x < x^2$; therefore, using the Mean Value Theorem on the interval $[x, x^2]$, there exists at least one value c_1 such

$$\text{that } \frac{\sin x^2 - \sin x}{x^2 - x} = \cos c_1 \text{ and } x < c_1 < x^2.$$

As $x \rightarrow 0^-$, $x^2 \rightarrow 0$; therefore, $c_1 \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^-} \frac{\sin x - \sin x^2}{x - x^2} &= \lim_{x \rightarrow 0^-} \frac{\sin x^2 - \sin x}{x^2 - x} \\ &= \lim_{c_1 \rightarrow 0} \cos c_1 \\ &= 1 \end{aligned}$$

Limits of Functions
and Their Relationships
(N121)

Since $x \rightarrow 0^+$, consider $0 < x < 1$.

(ii) When $x > 0$, consider $0 < x < 1$. Then, $0 < x^2 < x$; therefore, using the Mean Value Theorem on the interval $[x^2, x]$, there exists at least one value c_2 such that

$$\frac{\sin x - \sin x^2}{x - x^2} = \cos c_2 \text{ and } x^2 < c_2 < x.$$

As $x \rightarrow 0^+$, $x^2 \rightarrow 0$; therefore, $c_2 \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^+} \frac{\sin x - \sin x^2}{x - x^2} &= \lim_{c_2 \rightarrow 0} \cos c_2 \\ &= 1 \end{aligned}$$

Limits of Functions
and Their Relationships
(N121)

From (i) and (ii), $\lim_{x \rightarrow 0} \frac{\sin x - \sin x^2}{x - x^2} = 1$

$$\lim_{x \rightarrow 0} \frac{\sin x - \sin x^2}{x - x^2} = \lim_{x \rightarrow 0} \frac{\sin x - \sin x^2}{x - x^2}$$

Various Properties of Derivatives

Name _____

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1. Examine if the following function is continuous and also differentiable at $x=0$.

$$f(x) = |x|(x+2)$$

➡ N192

[Sol] When $x \geq 0$, $f(x) = x(x+2)$

When $x < 0$, $f(x) = -x(x+2)$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x(x+2) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} [-x(x+2)] = 0$$

Also, since $f(0) = 0$, $\lim_{x \rightarrow 0} f(x) = f(0)$

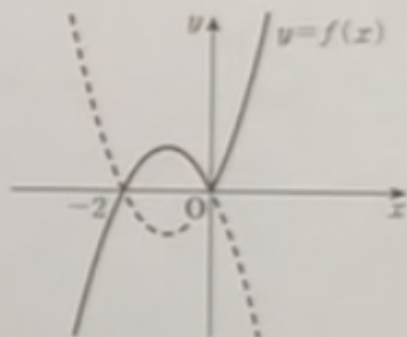
Therefore, $f(x)$ is **continuous** at $x=0$.

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(0+h)[(0+h)+2] - 0}{h} = \lim_{h \rightarrow 0^+} (h+2) = 2$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-(0+h)[(0+h)+2] - 0}{h} = \lim_{h \rightarrow 0^-} [-(h+2)] = -2$$

Thus, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Therefore, $f(x)$ is **not differentiable** at $x=0$.



2. Given that the function $f(x)$ is differentiable at $x=a$, express the following limit value in terms of $f'(a)$.

➡ N195

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a+h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a) + f(a) - f(a+h)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(a+2h) - f(a)}{2h} \cdot 2 - \frac{f(a+h) - f(a)}{h} \right] \\ &= 2f'(a) - f'(a) \\ &= f'(a) \end{aligned}$$

3. Find the value of c that satisfies the Mean Value Theorem for the following function and closed interval. ➡ N197

$$f(x) = \sqrt{2x-1} \quad [1, 5]$$

[Sol] $\frac{f(5)-f(1)}{5-1} = \frac{3-1}{4} = \frac{1}{2}$

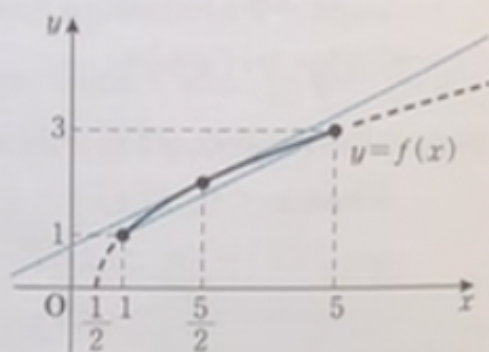
$$f'(x) = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x-1}}$$

From the Mean Value Theorem, there exists at least one value c such that

$$\frac{1}{2} = \frac{1}{\sqrt{2c-1}} \text{ and } 1 < c < 5.$$

$$2c-1=4$$

$$\therefore c = \frac{5}{2}$$



4. Using the Mean Value Theorem, prove the following inequality.

$$\text{When } a < b, e^a < \frac{e^b - e^a}{b-a} < e^b$$

➡ N198

[Sol] Let $f(x) = e^x$.

$f(x)$ is differentiable for all real numbers x and $f'(x) = e^x$.

Using the Mean Value Theorem on the interval $[a, b]$, there exists at least one value c such that $\frac{e^b - e^a}{b-a} = e^c$ and $a < c < b$.

Since $a < c < b$, $e^a < e^c < e^b$

$$\therefore e^a < \frac{e^b - e^a}{b-a} < e^b$$

Mathematics **Solution Book N**

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